THE UNIVERSITY OF ALBERTA

THE APPROXIMATE DISTRIBUTION OF THE SERIAL CORRELATION OF A PRESTATIONARY LINEAR MARKOV PROCESS

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE

OF MASTER OF SCIENCE

DEPARTMENT OF MATHEMATICS

by

SHAUKAT ABBAS

EDMONTON, ALBERTA

APRIL, 1964

ABSTRACT

The purpose of this thesis is to obtain the approximate distribution of an estimate of the serial correlation coefficient of a prestationary linear Markov process. The relevant literature is reviewed in Chapter I. Chapter II describes the prestationary linear Markov process, gives Daniels' development of the distributions of a sample serial correlation coefficient for the stationary linear Markov process with known and unknown means and reviews Patton's stationary distribution of an estimate of the serial correlation coefficient. In Chapter III, following Daniels' method, we obtain the approximate distribution of an estimate of the serial correlation coefficient when the mean is known and in Chapter IV for unknown mean. We also compare our distributions with those of Daniels' for $\rho = 0$.

ACKNOWLEDGEMENTS

I am indebted to Dr. J. R. McGregor for suggesting the problem. It is a pleasure to acknowledge his careful and considerate guidance during the course of this research.

I am also thankful to the Colombo Plan Authorities in Canada for financial support throughout the duration of this work.

TABLE OF CONTENTS

		Page
CHAPTER I	INTRODUCTION AND SUMMARY OF LITERATURE	1
CHAPTER II	THE APPROXIMATE DISTRIBUTIONS OF ESTIMATES OF THE SERIAL CORRELATION COEFFICIENT FOR A NON-CIRCULAR MARKOV PROCESS WITH KNOWN AND UNKNOWN MEANS	15
	1. Linear Markov Processes	15
	2. Non-Circular Markov Process: Daniels' Problem . .	19
	3. Non-Circular Markov Process: Patton's Case . . .	42
CHAPTER III	THE APPROXIMATE DISTRIBUTION OF ESTIMATES OF THE SERIAL CORRELATION COEFFICIENT FOR A PRESTATIONARY PROCESS WITH KNOWN MEAN	47
CHAPTER IV	DISTRIBUTION OF AN ESTIMATE OF THE SERIAL CORRELATION COEFFICIENT UNKNOWN MEAN	80
BIBLIOGRAPHY	. . . .	99
APPENDIX I	. . . .	104
APPENDIX II	. . . .	109
APPENDIX III	. . . .	119

CHAPTER I

INTRODUCTION AND SUMMARY OF LITERATURE

In the present work our object is to find an approximate distribution of a serial correlation coefficient in a pre-stationary linear Markov process. Though serial correlation has attracted many persons yet no one has so far taken up the problem in a pre-stationary case.

Serial correlation tests and distributions have been studied in various cases. Below we give a summary of some literature on the subject.

Anderson [1] defined the lag L serial correlation for N observations $X_1, X_2, \ldots, X_N$ by the expression

$$ {}_L R_N = \frac{X_1 X_{L+1} + \ldots + X_{N-L} X_N + X_N X_L - (\Sigma X_i)^2/N}{\Sigma X_i^2 - (\Sigma X_i)^2/N} \quad . $$

Under the assumption that $X_1, X_2, \ldots, X_N$ are independent $N(0,1)$ he derived the exact and large sample distributions of ${}_1R_N$ and gave a table of the exact significance points for all values of $N \leq 70$. For $N > 70$ the large sample approximation could be used. He proved that the distributions of ${}_LR_N$ and ${}_1R_N$ were the same if L and N were prime to each other and stated that the distribution of ${}_LR_N$ could be obtained for any L and N if it were known for values of L and N for which L was a factor of N. He also derived the distribution of ${}_LR_N$ and gave the significance points in the cases when $\frac{N}{L} = 2, 3$ and 4.

Koopmans [19] considered the stochastic process

$$ x_t = \rho\, x_{t-1} + z_t \qquad (t=1,2,3,\ldots) $$

where z_t were independent drawings from a normal distribution with mean zero and variance σ^2 and the coefficient ρ an unknown constant such that $|\rho| < 1$. On the basis of the observed values $x_1, x_2, \ldots, x_T$, the maximum likelihood estimate $\hat{\rho}$ taken as an estimate of ρ, was shown to be a function of the three quadratic forms

$$\ell = x_1^2 + x_T^2$$

$$m = x_1 x_2 + \ldots + x_{T-1} x_T$$

and $$n = x_2^2 + \ldots + x_{T-1}^2 \quad .$$

For testing the hypothesis that $\rho = 0$ the author showed that it was sufficient to know the distribution of $\frac{m}{\ell+n}$ and in this case studied the distribution of $r = \frac{q}{p}$ where

$$p = x_1^2 + x_2^2 + \ldots + x_T^2$$

and q is a quadratic form in the variables $x_1, x_2, \ldots, x_T$ with characteristic values $k_1, k_2, \ldots, k_T$. Assuming that the variates $x_1, x_2, \ldots, x_T$ are independent $N(0,1)$, he derived an expression for the probability density $h(r)$ of r taking $q = m$. He also considered the case

$$q = \bar{m} = m + x_T x_1$$

and showed that the characteristic values of m and $\bar{m}$ were

$$k_t = \cos \frac{\pi t}{T+1}$$

$$\bar{k}_t = \cos \frac{2\pi t}{T}$$

respectively. He obtained an approximate formula for the probability density of $\bar{r} = \frac{\bar{m}}{p}$ as

$$\tfrac{1}{2}(T-1)2^{\frac{T}{2}}\,\pi^{-1}\int_0^{\arc\cos\bar{r}}(\cos\alpha-\bar{r})^{\frac{T}{2}-2}\sin\tfrac{1}{2}T\alpha\sin\alpha\,d\alpha\ .$$

Dixon [6] considered distributions related to the quantities

$$\delta_n^2 = \sum_1^n (x_i - x_{i+1})^2 \ ,$$

$$C_n = \sum_1^n (x_i - \bar{x})(x_{i+1} - \bar{x}) \ ,$$

$$_{\ell}C_n = \sum_1^n (x_i - \bar{x})(x_{i+\ell} - \bar{x}) \ ,$$

$$v_n = \sum_1^n (x_i - \bar{x})^2$$

where $x_{n+i} = x_i$, the x's being independent $N(a,\sigma^2)$ and ℓ being the lag. Using the relationship

$$x_\alpha = a + bx_{\alpha-\ell}$$

the likelihood criterion to test

(a) H_1 : $b = 0$, was found to be

$$\lambda_1 = (1-\hat{b}^2)^{\frac{n}{2}} \ , \quad \text{where } \hat{b} = {}_{\ell}C_n/v_n$$

and

(b) $\quad {}_0H_1(a=0):\ b = 0$, was found to be

$$ {}_0\lambda_1 = (1-\hat{b}_0^2)^{\frac{n}{2}} , \quad \text{where } \hat{b}_0 = \frac{\Sigma x_\alpha x_{\alpha+\ell}}{\Sigma x_\alpha^2} . $$

Approximations to the moment generating function were used to derive the moments of $\hat{b}$ and $\hat{b}_0$ which were found to be exact for moments less than $\frac{2n}{\alpha}$, where α is the greatest common factor of ℓ and n. These moments were used to derive the distribution functions of $\hat{b}$, $\hat{b}_0$, $\lambda_1^{2/n}$ and ${}_0\lambda_1^{2/n}$. For the distribution of $\hat{b}$, a Pearson Type I approximation was used to determine 1% and 5% positive and negative significance levels. The moments and distribution were also obtained for

$$ {}_0\eta_1 = \frac{\delta_n^2}{\Sigma x_i^2} = 2(1-\hat{b}_0) $$

and

$$ \eta_1 = \frac{\delta_n^2}{v_n} $$

and the first 2 moments for

$$ \eta_2 = \sum_1^n (x_i - 2x_{i+1} + x_{i+2})^2 / v_n . $$

Dixon also set up the general λ-criterion to test

$$ H_{r,m}:\quad b_{m+1}, b_{m+2}, \ldots, b_r = 0 \quad \text{for} \quad x_\alpha = a + \sum_1^r b_i x_{\alpha-\ell_i} $$

and gave the mean and variance for $r = 2$ and $m = 0, 1$ $(a = 0$, and $a \neq 0)$. He also indicated the set-up, without solution, for the serial correlation in several variables.

Rubin [28] proved the equality of approximations given by Koopmans [19] and Dixon [6] to the distribution of the serial correlation coefficient $\bar{r}$, assuming that the true value ρ of $\bar{r}$ was zero, where

$$\bar{r} = \sum_{t=1}^{T} \chi_t \chi_{t+1} \Big/ \sum_{t=1}^{T} \chi_t^2 ,$$

$\chi_{T+1} = \chi_1$ and χ_t are independent $N(0,\sigma^2)$.

Madow [22] extended the results of Anderson [1] and derived the distribution of the serial correlation coefficient, using the circular definition, for the population value $\rho \neq 0$.

Leipnik [20] considered the circular sequence of random variables $x_0, x_1, \ldots, x_T = x_0$ such that the differences

$$z_t = x_t - \rho x_{t-1}$$

$(|\rho| < 1,\ t = 1,2,\ldots,T)$ were independent and normal with zero means and equal variances σ^2. As estimates for σ^2 and ρ the expressions $\frac{p}{T}$ and $\bar{r} = \frac{\bar{q}}{p}$ respectively were used, where

$$p = x_1^2 + x_2^2 + \ldots + x_T^2$$

and $$\bar{q} = x_1 x_2 + \ldots + x_T x_1 .$$

By smoothing the joint characteristic function of p and $\bar{q}$, he deduced appropriate expressions for the frequency functions of p and $\bar{r}$ in the case of an arbitrary ρ. For $\bar{r}$ he obtained the frequency function

$$\tilde{R}_\rho(\bar{r}) = \frac{\Gamma(\frac{1}{2}T+1)}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}T+\frac{1}{2})} (1-\bar{r}^2)^{\frac{1}{2}(T-1)} (1+\rho^2-2\rho\bar{r})^{-\frac{1}{2}T} .$$

He exhibited the graphs of this distribution for $T = 20$ and various values of ρ. He also obtained general expressions for the approximate moments of p and $\bar{r}$ and explicitly evaluated the mean and variance of these variables.

Anderson and Anderson [2] tested the independence of z_t, which is the normal disturbance in a seasonal variation of a time series x_t, against the hypothesis of linear regression of the type

$$z_t = \rho \, z_{t-L} + u_t \qquad (\rho \neq 0)$$

by the likelihood ratio criterion which is shown to be equivalent to the circular coefficient R_L. They also obtained exact and approximate distributions of R_L and tabulated its significance points for $L = 1$.

Ogawara [24] used normal regression theory to estimate the parameters in a stationary Gaussian Simple Markov Process under the assumption that alternate values of the process were fixed, set up the confidence limits for the autocorrelation coefficient and indicated the theory for a process of order h.

Moran [23] suggested testing serial correlation among the residuals ϵ_i from the regression line to make sure that the apparent serial correlation of the dependent variable may not be entirely due to serial correlation of the independent variable. He considered the first-order cyclic serial correlation coefficient

$$R_1 = \frac{\sum_1^n \epsilon_i \epsilon_{i+1}}{\sum_1^n \epsilon_i^2} \quad ,$$

where $\epsilon_{n+1} = \epsilon_1$, and derived $E(R_1)$, the expected value of R_1, and $V(R_1)$, the variance of R_1, under the assumption that the true residuals

are independently distributed in the same normal distribution and stated that

$$\frac{R_1 - E(R_1)}{\sqrt{V(R_1)}}$$

is asymptotically normally distributed.

Watson and Durbin [32] presented an exact non-circular test of the existence of serial correlation in a series of n observations.

Quenouille [26] derived the joint distribution of circularly defined serial correlation coefficients on the assumptions that the observations were normally and independently distributed. The results could be extended to the case where the observations satisfied the relation

$$a_0 x_i + a_1 x_{i-1} + \ldots + a_m x_{i-m} = \epsilon_i$$

where $\epsilon_{i+n} = \epsilon_i$ and the ϵ's are normally and independently distributed. Since the distributions were difficult to compute, only approximate forms were investigated.

Gibson [12], in an investigation which involved matching a series of tree-ring width measurements, constructed a routine method for the rapid measurement of the lagged correlations between two series. Instead of the usual second order moment correlation he used a first order moment correlation based on absolute values of sums and differences.

Durbin and Watson [8] taking

$$Y = X\,\beta + \epsilon$$

where Y was the n-component column vector of dependent variates, X the

(n x k) vector of independent (fixed variates), β the vector of regression coefficients and ϵ the vector of (unobservable) random variables, tested the hypothesis that the components of ϵ were uncorrelated against the hypothesis that they were serially correlated. They used the statistic

$$r = \frac{z'Az}{z'z}$$

where A was some given positive definite matrix and $z = My$ was the vector of residuals of an observed vector y from the fitted regression on X and $M = I - X(X'X)^{-1}X$. The ratio of quadratic forms

$$r = y'M'AMy \,/\, y'MMy$$

could be reduced to

$$\sum_{i=1}^{n-k} \nu_i \zeta_i^2 \Bigg/ \sum_{i=1}^{n-k} \zeta_i^2 \qquad (\nu_1 \leq \nu_2 \leq \cdots \leq \nu_{n-k})$$

by an orthogonal transformation $y = H\zeta$. If the components of ϵ were independently $N(0,\sigma^2)$ it was shown that $\zeta_1, \zeta_2, \ldots, \zeta_{n-k}$ were also independently $N(0,\sigma^2)$ and the inequalities on the distribution of r were obtained by obtaining the inequalities on $\nu_1, \nu_2, \ldots, \nu_{n-k}$ in terms of the latent roots of A. In [9] upper and lower bounds to the 5%, 2.5% and 1% significance points of the distribution function of the criterion in [8] were presented for 1 to 5 fixed variates in the regression model and for 15 to 100 observations and an approximate procedure of testing was also outlined when the bounds test was inclusive.

Sastry [29] gave the expressions for the expected value of the bias in the empirical autocorrelation coefficient caused by the correction for the mean and obtained some numerical results in some representative cases.

Walker [31] considered $n + u$ $(u \geq 0)$ consecutive observations $X_1, X_2, \ldots, X_{n+u}$, from the stationary autoregressive process

$$\sum_{i=o}^{p} \alpha_i X_{t-i} = Y_t$$

where $\{Y_t\}$ is a stationary m-dependent process with $E(Y_t) = 0$ and $E(Y_t^4) < \infty$ and showed that the distribution function of

$$Z_{n,u} = n^{-\frac{1}{2}} \sum_{t=1}^{n} \{X_t X_{t+u} - E(X_t X_{t+u})\}$$

tends to that of a normal variate with zero mean and finite variance.

Jenkins [17] proposed to stabilise the variance of the first auto-correlation coefficient r (of a sample series generated by a first-order autoregression with $E(r) = \rho$) by the use of $\sin^{-1}(r)$ and showed that this variate was asymptotically normal, more stable than $\tanh^{-1}(r)$ and useful for test purposes when $|\rho| < 0.9$.

Rao [27] considered the equation

$$x_{st} = E(x_{st}) + \epsilon_{st} , \qquad 1 \leq s \leq k, \quad 1 \leq t \leq n$$

where $E(x_{st})$ is a linear function of lagged values of k variables in a simultaneous equation model with k endogenous variables and n time periods and tested the hypothesis that the n normal vectors $[\epsilon_t]$ are independent against a wide class of alternatives, where

$$\epsilon_t' = [\epsilon_{1t}, \epsilon_{2t}, \ldots, \epsilon_{st}, \ldots \epsilon_{kt}] .$$

Hannan [14] showed that Ogawara's [24] exact test was asymptotically fully efficient if the parent scheme is a first order autoregression and extended this test to give an exact test of independence of residuals from a regression equation with given values of the regression coefficients.

Watson [33] considered the consequences of assuming an incorrect covariance matrix for the residuals from regression of a set of observations and deduced inequalities on the bias in the estimates of variance of the regression coefficients, on the efficiency of the estimates of the regression coefficients and on the significance points of the various t- and F- tests.

Daniels [5] applied his adaptation of the method of steepest descent to improve upon earlier results especially with regard to the order of approximation and to treat more general cases. He derived distributions for the first order serial coefficient in a stationary Gaussian Markov process in the circular and non-circular cases and for the joint distribution of the first m ordinary and partial coefficients in circular autoregressive process of order m. He considered both the cases of known and unknown means. He obtained the interesting result that, with a Markov Process, Leipnik's [20] formula for the circular case needs only a change in the degrees of freedom to apply in the non-circular case, with an error of order $O(T^{-3/2})$ where T is the number of observations.

Watson [34], using the characteristic function and latent root representation of the serial coefficients, derived the approximate joint distribution for the case of a Gaussian process with zero autocorrelations and non-zero means and compared this with more precise results by Jenkins [17] and Daniels [5].

Hannan [15] extended the tests for Serial Correlation [14] to vector Markov Processes and tested the independence of successive vector variates μ_t by examining the canonical correlation between μ_{2t} and (μ_{2t-1}, μ_{2t+1}). He also extended the method to provide an exact test for the serial correlation of a vector or residuals from regression and obtained bounds for the asymptotic efficiencies of the tests. He [16] also obtained the asymptotic distribution of a statistic r to be used in testing for serial correlation in the residuals of a regression of the form $Y = X\beta + \epsilon$ with the components of ϵ generated by a stationary process. He also found approximate lower bounds for percentage points of the distribution of r when the regression was on orthogonal polynomials or on orthogonal polynomials and random variables.

White [37] considered the distribution

$$f(t) = C\,(1-2\rho t+\rho^2)^{-\frac{N}{2}}\,(1-t^2)^{\frac{N-1}{2}} \qquad (-1 \leq t \leq 1)$$

which is an approximation to that of the serial-correlation coefficient of a circular first order Gaussian autoregressive process and derived formulas for its moments.

Durbin [7] extended the results of Durbin and Watson [8 and 9] to models with $p + 1$ interrelated equations in $p + 1$ dependent or endogenous variables (y) and $k + q$ $(q \geq p$ and $k \geq 0)$ independent or exogenous variables (x). Taking simultaneous equations of the form

$$Ay = Bx + \epsilon$$

where ϵ represented the vector of errors or disturbances and assuming that n simultaneous observations were obtained on the variables and that q of

the elements of the corresponding vector of B were zero, results to test for serial dependence in a given equation of the system were obtained.

Leipnik [21] used a method of Kac and Kac and Erdős to obtain a closed form for the joint generating function of

$$X_1^2 , \sum_2^n X_j^2 , X_{n+1}^2 , \sum_1^n X_j X_{j+1}$$

where the X_j's are serially correlated normal variables. He also computed the exact moments of several estimates of the variance and autocorrelation in the case of the Ornstein-Uhlenbeck process.

Siddiqui [30] showed that for the first order autocorrelation coefficient r of N independent $N(0,1)$ variables, the expansion of Prob $(r > r_0)$ in powers of $1 - r_0$ begins with

$$C(1 - r_0)^{-\frac{(N-2)}{2}} , \qquad 0.14 < C < 0.22 .$$

Weinstein [36] tested various estimators for the serial correlations of small lags in a stationary series with an unknown mean and variance using an artificially constructed second order autoregressive process with samples of 14 and showed that the result of eliminating the unknown mean was more crucial than eliminating the unknown variance.

White [38] showed that the limiting distribution (suitably normalised) of the least squares estimate $\hat{\alpha}$ of the autoregression was Cauchy when $|\alpha| > 1$, normal when $|\alpha| < 1$ and of a third form whose characteristic function was obtained by him when $|\alpha| = 1$. He [39] further showed that for any α, with the possible exception of the case $|\alpha| = 1$, the statistic

$$W = \frac{\hat{\alpha} - \alpha}{\hat{\sigma}} \left(\sum_{1}^{T} x_{t-1}^2 \right)^{\frac{1}{2}}$$

is asymptotically $N(0,1)$ where $\hat{\alpha}$ and $\hat{\sigma}$ are the maximum likelihood estimates of the parameters of the autoregression

$$x_t = \alpha\, x_{t-1} + u_t$$

the u's being independent $N(0,\sigma^2)$. He [40] also obtained expansions for the expectation and variance of the maximum likelihood estimate, $\hat{\alpha}$, of the serial correlation coefficient, α, of a first order autoregressive Gaussian process, as far as the terms in T^{-3} and α^4 (where T is the length of the series) for the case where the first observation was unconditional and was known to be equal to zero.

Griliches [13] considered the model

$$y_t = \beta\, x_t + \gamma\, y_{t-1} + e_t \; ; \quad e_t = \rho\, e_{t-1} + w_t$$

where w_t are independently distributed random variables with zero means and y_t and e_t are stationary stochastic processes and showed that the simple least squares estimator c of γ was biased and that the bias was positive. He further showed that if $\gamma = 0$ but the estimating equation

$$\hat{y}_t = b\, x_t + c\, y_{t-1}$$

was used to reduce the serial correlation of the residuals, the expected value of c would be

$$\rho(1 - r_{yx}^2) \quad .$$

Chanda [3] proved that the maximum of the absolute value of the s^{th} order serial correlation

$$r_s = \frac{n}{n-s} \left\{ \sum_{1}^{n-s} x_t \, x_{t+s} \Big/ \sum_{1}^{n} x_t^2 \right\} ,$$

was

$$\frac{n}{n-s} \cos \frac{\pi}{m+2} \qquad \text{for} \quad n = ms + u \ , \quad u > 0$$

and

$$\frac{m}{m-1} \cos \frac{\pi}{m+1} \qquad \text{for} \quad n = ms \ .$$

In the present thesis we consider the linear Markov process

$$x_s = \rho^{s-1} e_1 + \rho^{s-2} e_2 + \ldots + \rho \, e_{s-1} + e_s$$

where, $e_1, e_2, \ldots, e_s$ are independent $N(0,1)$ random variables and ρ (the serial correlation coefficient) is such that $|\rho| < 1$. We review Daniels' method and work in obtaining the approximate distribution of the sample serial correlation coefficient

$$r = \frac{c}{c_0}$$

where

$$c = x_1 x_2 + \ldots + x_{n-1} x_n$$

$$c_0 = \tfrac{1}{2} x_1^2 + x_2^2 + \ldots + x_{n-1}^2 + \tfrac{1}{2} x_n^2$$

and then use Daniels' method to obtain the approximate distribution of r in the prestationary linear Markov process under consideration. We obtain the distribution of r in the case of known as well as the unknown mean.

CHAPTER II

THE APPROXIMATE DISTRIBUTIONS OF ESTIMATES OF THE SERIAL CORRELATION COEFFICIENT FOR A NON-CIRCULAR MARKOV PROCESS WITH KNOWN AND UNKNOWN MEANS

1. Linear Markov Processes

Consider a subset of time points

$$t_1 < t_2 < \dots < t_n$$

and the random variables $X(t_j)$ at time t_j. The joint distribution of $X(t_1), \dots, X(t_n)$ is denoted by $p[X(t_1), \dots, X(t_n)]$. We have a completely stationary process if

$$p[X(t_1),\dots,X(t_n)] = p[X(t_1-\tau),\dots,X(t_n-\tau)]$$

for all subsets $t_1 < t_2 < \dots < t_n$. This means that the joint distribution is invariant under a change of time origin. A process is stationary of order k if

$$\mathcal{E}[X^{\alpha_1}(t_1) \dots X^{\alpha_n}(t_n)] = \mathcal{E}[X^{\alpha_1}(t_1-\tau) \dots X^{\alpha_n}(t_n-\tau)]$$

for all τ and all $\alpha_1 + \alpha_2 + \dots + \alpha_n \leq k$.

We shall now consider the cases of order one and two. If the process is stationary of order one,

$$\mathcal{E}[X(t)] = \mathcal{E}[X(t-\tau)] = \mu$$

for all τ which shows that the mean is independent of time. If the process is stationary of order two, we have

$$\mathcal{E}[X^2(t)] = \mathcal{E}[X^2(t-\tau)]$$

for all τ, and

$$\mathcal{E}[X(t_1)X(t_2)] = \mathcal{E}[X(t_1-\tau)X(t_2-\tau)]$$

for all τ. Then

$$\begin{aligned}\mathrm{Var}[X(t)] &= \mathcal{E}[X^2(t)] - \{\mathcal{E}[X(t)]\}^2 \\ &= \mathcal{E}[X^2(t-\tau)] - \{\mathcal{E}[X(t-\tau)]\}^2 \\ &= \mathrm{Var}[X(t-\tau)] \\ &= \sigma^2\end{aligned}$$

for all τ. This shows that the variance is independent of time. We call σ^2 the autovariance. Also

$$\begin{aligned}\mathrm{Cov}[X(t_1),X(t_2)] &= \mathcal{E}[X(t_1)X(t_2)] - \mathcal{E}[X(t_1)]\,\mathcal{E}[X(t_2)] \\ &= \mathcal{E}[X(t_1-\tau)X(t_2-\tau)] - \mathcal{E}[X(t_1-\tau)]\,\mathcal{E}[X(t_2-\tau)] \\ &= \mathrm{Cov}[X(t_1-\tau),\ X(t_2-\tau)] \\ &= C(t_2-t_1)\end{aligned}$$

for all τ. $C(t_2-t_1)$ is called the autocovariance and, as shown, is independent of the time origin.

The autocorrelation function $\rho(\tau')$ is defined as:

$$\frac{C(t_2-t_1)}{\sigma^2} = \rho(\tau')$$

where $\tau' = t_2 - t_1$ is the lag. $\rho(\tau')$ measures the degree to which variables at time difference τ' are correlated. $\rho(\tau')$ is an even function of τ'. It can also be shown that necessary and sufficient conditions for a function to be an autocorrelation function are that

$$\rho(o) = 1 , \quad \rho(\tau') \text{ is even}$$

and

$$\rho(\tau') \text{ is positive semi-definite.}$$

A process which is stationary of order one but not stationary of order two will be called a pre-stationary process.

The linear Markov process in discrete time is defined by

(2.1.1) $$x_s = \rho\, x_{s-1} + e_s \qquad (s = 1,2,\ldots,n)$$

where the constant ρ is called the serial correlation coefficient and the $\{e_s\}$ are independently and identically distributed, $N(0,1)$ random variables. Following Patton [(25),pages9-11] we have

$$\mathcal{E}(x_s) = \rho^s\, x_o \rightarrow o$$

where x_o is the initial state,

$$\mathrm{Var}(x_s) = \frac{1 - \rho^{2s}}{1 - \rho^2} \rightarrow \frac{1}{1 - \rho^2} ,$$

and

$$\mathrm{Cov}(x_s, x_{s-\tau}) \rightarrow \frac{\rho^{|\tau|}}{1 - \rho^2}$$

for all τ if $|\rho| < 1$ and $s \rightarrow \infty$.

Hence if $|\rho| < 1$, the process ultimately becomes stationary to the second order, which in the case of a normal process implies complete stationarity.

Now let us consider a linear Markov process in discrete time defining x_i $(i = 1,2,\ldots,n)$ as follows:

$$(2.1.2)\qquad \begin{cases} x_1 = e_1 \\ x_2 = \rho e_1 + e_2 \\ x_3 = \rho^2 e_1 + \rho e_2 + e_3 \\ \dots\dots\dots\dots\dots \\ x_n = \rho^{n-1} e_1 + \rho^{n-2} e_2 + \dots + \rho e_{n-1} + e_n \end{cases}$$

where e_i $(i = 1,2,\dots,n)$ are independent $N(0,1)$ random variables and $|\rho| < 1$.

We observe that for $s = 1,2,\dots,n$,

$$x_s = \rho^{s-1} e_1 + \rho^{s-2} e_2 + \dots + \rho^2 e_{s-2} + \rho e_{s-1} + e_s \quad .$$

Since

$$\mathcal{E}(e_i) = 0 , \qquad (i = 1,2,\dots,n),$$

$$\mathcal{E}(x_s) = 0 \qquad (s = 1,2,\dots,n) \quad .$$

Hence the process is stationary to the first order.

Again since

$$\mathcal{E}(e_i^2) = 1 \qquad (i = 1,2,\dots,n),$$

and

$$\mathcal{E}(e_i e_j) = 0 \qquad (i \neq j),$$

we have

$$\mathcal{E}(x_s^2) = \rho^{2(s-1)} + \rho^{2(s-2)} + \dots + \rho^4 + \rho^2 + 1$$

$$= \frac{1 - \rho^{2s}}{1 - \rho^2} \quad ,$$

and

$$\mathcal{E}(x_{s-\tau}^2) = \frac{1 - \rho^{2s-2\tau}}{1 - \rho^2} .$$

Therefore

$$\mathcal{E}(x_s^2) \neq \mathcal{E}(x_{s-\tau}^2) .$$

Again

$$\mathcal{E}(x_s\, x_t) = \rho^{s+t-2} + \rho^{s+t-4} + \ldots + \rho^{s-t} \qquad (s > t)$$

$$= \rho^{s-t} (\rho^{2t-2} + \rho^{2t-4} + \ldots + \rho^2 + 1)$$

$$= \rho^{s-t} \left(\frac{1 - \rho^{2t}}{1 - \rho^2} \right)$$

and

$$\mathcal{E}(x_{s-\tau}\, x_{t-\tau}) = \rho^{s-t} \left(\frac{1 - \rho^{2t-2\tau}}{1 - \rho^2} \right).$$

Hence

$$\mathcal{E}(x_s\, x_t) \neq \mathcal{E}(x_{s-\tau}\, x_{t-\tau}) .$$

Therefore the process defined in equations (2.1.2) is a pre-stationary one.

2. <u>Non-circular Markov Process: Daniels' Problem.</u>

Consider the process $x_s = \rho\, x_{s-1} + e_s$ for all s where the e's are independent $N(0,1)$ random variables. We assume that $|\rho| < 1$ and that the process is stationary.

Let $x_1, x_2, \ldots, x_n$ be a sample of observations from the stationary process. They have a joint multivariate normal distribution since $e_1, e_2, \ldots, e_n$ are independent, $N(0,1)$ random variables. The variance covariance matrix of $x_1, x_2, \ldots, x_n$ is the $(n \times n)$ matrix

$$(2.2.1)\qquad \underline{C} = \begin{bmatrix} \frac{1}{1-\rho^2} & \frac{\rho}{1-\rho^2} & \cdots & \frac{\rho^{n-1}}{1-\rho^2} \\ \frac{\rho}{1-\rho^2} & \frac{1}{1-\rho^2} & \cdots & \frac{\rho^{n-2}}{1-\rho^2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\rho^{n-1}}{1-\rho^2} & \frac{\rho^{n-2}}{1-\rho^2} & & \frac{1}{1-\rho^2} \end{bmatrix}_n .$$

Multiplying the second column by ρ and subtracting it from the first column in $|\underline{C}|$ and repeating the same procedure for third and second columns and so forth, we get

$$(2.2.2)\qquad |\underline{C}| = \frac{1}{1-\rho^2} .$$

It is easily shown that $\underline{C}^{-1}$ is the $(n\text{x}n)$ matrix

$$(2.2.3)\qquad \underline{C}^{-1} = \begin{bmatrix} 1 & -\rho & & & & (0) & \\ -\rho & 1+\rho^2 & -\rho & & & & \\ & -\rho & 1+\rho^2 & -\rho & & & \\ & & & \cdots\cdots & & & \\ & (0) & & & -\rho & 1+\rho^2 & -\rho \\ & & & & & -\rho & 1 \end{bmatrix} .$$

The joint distribution of $x_1, x_2, \ldots, x_n$ is given by

$$(2.2.4)\qquad dF(x_1,x_2,\ldots,x_n) = \frac{1}{(2\pi)^{\frac{n}{2}}} \cdot \frac{1}{|\underline{C}|^{\frac{1}{2}}} \exp\{-\tfrac{1}{2}\, \underline{x}'\, \underline{C}^{-1}\, \underline{x}\}\, dx_1 \ldots dx_n ,$$

where $\underline{C}$ is the variance-covariance matrix and

$$\underline{x}' = (x_1, x_2, \ldots, x_n) .$$

For the case under consideration, where $|\underline{C}|$ and $\underline{C}^{-1}$ are given by equations (2.2.2) and (2.2.3) respectively, we have

$$dF(x_1,\ldots,x_n) = \frac{(1-\rho^2)^{\frac{1}{2}}}{(2\pi)^{\frac{n}{2}}} \exp\{-\tfrac{1}{2}[x_1^2+(1+\rho^2)(x_2^2+\ldots+x_{n-1}^2)+x_n^2-2\rho(x_1x_2+\ldots+x_{n-1}x_n)]\} \, dx_1\ldots dx_n .$$

The sample estimate of ρ is taken to be

(2.2.5) $$r = \frac{c}{c_o}$$

where

(2.2.6) $$c = x_1x_2 + x_2x_3 + \ldots + x_{n-1}x_n ,$$

and

(2.2.7) $$c_o = \tfrac{1}{2}x_1^2 + x_2^2 + \ldots + x_{n-1}^2 + \tfrac{1}{2}x_n^2 .$$

The joint Moment Generating Function of c and c_o is

$$M(T_o,T) = \mathcal{E}(e^{T_oc_o+Tc})$$

$$= \frac{(1-\rho^2)^{\frac{1}{2}}}{(2\pi)^{\frac{n}{2}}} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{T_oc_o+Tc-\frac{1}{2}\underline{x}'\underline{C}^{-1}\underline{x}} \, dx_1dx_2\ldots dx_n .$$

Now

$$T_o c_o + Tc - \tfrac{1}{2}\, \underline{x}'\underline{C}^{-1}\underline{x} = -\tfrac{1}{2}[(1-T_o)x_1^2 + (1+\rho^2-2T_o)(x_2^2+\ldots+x_{n-1}^2)$$

$$+(1-T_o)x_n^2 - 2(\rho+T)(x_1x_2+\ldots+x_{n-1}x_n)]$$

$$= -\tfrac{1}{2}\, \underline{x}'\, \underline{B}\, \underline{x}$$

where $\underline{B}$ is the $(n\text{x}n)$ matrix

$$(2.2.8) \qquad \underline{B} = \begin{bmatrix} 1-T_o & -(\rho+T) & & & & & (0) \\ -(\rho+T) & 1+\rho^2-2T_o & -(\rho+T) & & & & \\ & -(\rho+T) & 1+\rho^2-2T_o & -(\rho+T) & & & \\ & & & \cdots\cdots\cdots & & & \\ & (0) & & & -(\rho+T) & 1+\rho^2-2T_o & -(\rho+T) \\ & & & & & -(\rho+T) & 1-T_o \end{bmatrix}$$

Then $M(T_o,T)$ may be written as

$$M(T_o,T) = \frac{(1-\rho^2)^{\frac{1}{2}}}{(2\pi)^{\frac{n}{2}}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\frac{1}{2}\underline{x}'\,\underline{B}\,\underline{x}}\, dx_1 \ldots dx_n \quad .$$

There is an orthogonal transformation that will diagonalise $\underline{B}$. Making this transformation and performing the integration we get

$$(2.2.9) \qquad M(T_o,T) = \frac{(1-\rho^2)^{\frac{1}{2}}}{|\underline{B}|^{\frac{1}{2}}} \; ,$$

where $|\underline{B}|^{-\frac{1}{2}}$ is the Jacobian of the transformation.

From equation (2.2.8) and using (II.20) and (II.22) we observe that

$$|\underline{B}| = f\left[\frac{(p-\beta)f - q^2}{\alpha - \beta}\,\alpha^{n-2} + \frac{(\alpha-p)f + q^2}{\alpha - \beta}\,\beta^{n-2}\right]$$

$$- q^2\left[\frac{(p-\beta)f - q^2}{\alpha - \beta}\,\alpha^{n-3} + \frac{(\alpha-p)f + q^2}{\alpha - \beta}\,\beta^{n-3}\right] ,$$

where

$$(2.2.10)\qquad \begin{cases} \alpha + \beta = p , \\ \alpha\beta = q^2 , \\ p = 1 + \rho^2 - 2T_o , \\ q = -(\rho + T) \qquad \text{and} \\ f = (1 - T_o) . \end{cases}$$

Substituting for p and q in terms of α and β we get

$$|\underline{B}| = f\left[\frac{\alpha f - \alpha\beta}{\alpha - \beta}\,\alpha^{n-2} + \frac{\alpha\beta - \beta f}{\alpha - \beta}\,\beta^{n-2}\right]$$

$$- \alpha\beta\left[\frac{\alpha f - \alpha\beta}{\alpha - \beta}\,\alpha^{n-3} + \frac{\alpha\beta - \beta f}{\alpha - \beta}\,\beta^{n-3}\right]$$

or

$$(2.2.11)\qquad |\underline{B}| = (f - \beta)^2\,\frac{\alpha^{n-1}}{\alpha - \beta} - (\alpha - f)^2\,\frac{\beta^{n-1}}{\alpha - \beta} .$$

Let

$$(2.2.12)\qquad z + \frac{1}{z} = \frac{1 + \rho^2 - 2T_o}{\rho + T} = -\frac{p}{q}$$

which gives us

$$(2.2.13)\qquad z^2 + \frac{p}{q}\,z + 1 = 0 .$$

The roots of this equation are

$$z = \frac{1}{2}\left[-\frac{p}{q} + \sqrt{\frac{p^2}{q^2} - 4}\right]$$

$$= -\frac{1}{q}\left[\frac{p - \sqrt{p^2 - 4q^2}}{2}\right]$$

$$= -\frac{\beta}{q} \quad ,$$

and

$$\frac{1}{z} = \frac{1}{2}\left[-\frac{p}{q} - \sqrt{\frac{p^2}{q^2} - 4}\right]$$

$$= -\frac{1}{q}\left[\frac{p + \sqrt{p^2 - 4q^2}}{2}\right]$$

$$= -\frac{\alpha}{q} \quad ,$$

or

$$\text{(2.2.14)} \qquad \begin{cases} \alpha = -\dfrac{q}{z} = \dfrac{\rho + T}{z} \\ \text{and} \\ \beta = -qz = (\rho + T)z \quad . \end{cases}$$

Substituting for p, q, α, β from equations (2.2.10) and (2.2.14) in equation (2.2.11) we get

$$\text{(2.2.15)} \quad |\underline{B}| = (1-T_o-\rho z+Tz)^2 \left(\frac{\rho+T}{z}\right)^{n-1} \frac{z}{(\rho+T)(1-z^2)}$$

$$-\left(\frac{\rho+T}{z} - 1 + T_o\right)^2 (\rho+T)^{n-1} z^{n-1} \frac{z}{(\rho+T)(1-z^2)} \quad .$$

Since

$$\frac{1 + \rho^2 - 2T_o}{\rho + T} = z + \frac{1}{z} = \frac{z^2 + 1}{z} \quad ,$$

$$(1 + \rho^2 - 2T_o)z = (\rho + T)(1 + z^2)$$

which gives

$$(1 - T_o)z + (\rho^2 - T_o)z = \rho + T + z^2(\rho + T)$$

which in turn gives us

$$1 - T_o - \frac{(\rho + T)}{z} = z(\rho + T) - (\rho^2 - T_o)$$

and

$$1 - T_o - z(\rho + T) = \frac{\rho + T}{z} - (\rho^2 - T_o) \quad .$$

Substituting for these values in equation (2.2.15) we get

$$|\underline{B}| = \left[\frac{\rho + T}{z} - (\rho^2 - T_o) \right]^2 \left(\frac{\rho+T}{z}\right)^{n-2} \frac{1}{(1-z^2)}$$

$$- \left[z(\rho+T) - (\rho^2-T_o) \right]^2 (\rho+T)^{n-2} \frac{z^n}{(1-z^2)}$$

or

$$(2.2.16) \qquad |\underline{B}| = \frac{(\rho+T)^n}{z^n(1-z^2)} \left[\left\{ 1 - \frac{z(\rho^2-T_o)}{\rho + T} \right\}^2 - \left\{ z - \frac{\rho^2-T_o}{(\rho+T)} \right\}^2 z^{2n} \right] \quad .$$

As mentioned by Daniels' [5] if $1-r^2$ is not small, leaving out the term in z^{2n} within bracket in equation (2.2.16) incurs an exponentially small error in the final approximation. Hence

$$(2.2.17) \qquad |\underline{B}| \sim \frac{(\rho+T)^n}{z^n(1-z^2)} \left[1 - \frac{z(\rho^2-T_o)}{(\rho+T)} \right]^2 \quad .$$

Now let

$$(2.2.18) \qquad u = T_o + r\,T$$

so that

$$z + \frac{1}{z} = \frac{1 + \rho^2 - 2T_o}{\rho + T} = \frac{1 + \rho^2 - 2u + 2rT}{\rho + T}$$

or

$$z + \frac{1}{z} = \frac{1 + \rho^2 - 2\rho r - 2u}{\rho + T} + 2r \ . \tag{2.2.19}$$

Hence

$$\frac{1 - 2rz + z^2}{z} = \frac{1 + \rho^2 - 2\rho r - 2u}{\rho + T}$$

and

$$\rho + T = \frac{z(1 + \rho^2 - 2\rho r - 2u)}{1 - 2rz + z^2} \ , \tag{2.2.20}$$

which gives

$$T = -\left\{ \frac{(1 - \rho z)(\rho - z) + 2uz}{1 - 2rz + z^2} \right\} \ , \tag{2.2.21}$$

and

$$T_o = \frac{(1 + z^2)(u + \rho r) - rz(1 + \rho^2)}{1 - 2rz + z^2} \ . \tag{2.2.22}$$

By use of equations (2.2.18), (2.2.20) and (2.2.21) we obtain

$$1 - \frac{z(\rho^2 - T_o)}{\rho + T} = 1 - \frac{(1 - 2rz + z^2)(\rho^2 - u + rT)}{(1 - 2\rho r + \rho^2 - 2u)}$$

$$= \frac{(1-2\rho r+\rho^2-2u) - (\rho^2-u)(1-2rz+z^2) + r[(1-\rho z)(\rho-z)+2uz]}{1 - 2\rho r + \rho^2 - 2u}$$

or

$$1 - \frac{z(\rho^2 - T_o)}{\rho + T} = \frac{(1 - \rho z)[1 - \rho r - (r - \rho)z] - u(1 - z^2)}{1 - 2\rho r + \rho^2 - 2u} \ . \tag{2.2.23}$$

Finally equations (2.2.20) and (2.2.23) together with equation (2.2.17) give us

$$|\underline{B}| \sim \frac{z^n(1-2\rho r+\rho^2-2u)^n}{z^n(1-z^2)(1-2rz+z^2)^n} \left\{ \frac{(1-\rho z)[1-\rho r-(r-\rho)z]-u(1-z^2)}{1-2\rho r+\rho^2-2u} \right\}^2$$

$$= \frac{(1-2\rho r+\rho^2-2u)^{n-2}}{(1-z^2)(1-2rz+z^2)^n} \{(1-\rho z)[1-\rho r-(r-\rho)z] - u(1-z^2)\}^2 \quad .$$

Substituting for $|\underline{B}|$ in (2.2.9) we get

$$\text{(2.2.24)} \quad M(u-rT,T) \sim \frac{(1-\rho^2)^{\frac{1}{2}}(1-z^2)^{\frac{1}{2}}(1-2rz+z^2)^{\frac{n}{2}}}{(1-2\rho r+\rho^2-2u)^{\frac{n}{2}-1}\{(1-\rho z)[1-\rho r-(r-\rho)z]-u(1-z^2)\}} \quad .$$

Differentiating equation (2.2.20) with respect to z, we get

$$\text{(2.2.25)} \qquad \frac{\partial T}{\partial z} = \frac{(1-z^2)(1-2\rho r+\rho^2-2u)}{(1-2rz+z^2)^2} \quad .$$

Combining equations (2.2.24) and (2.2.25) we have

$$M(u-rT,T)\frac{\partial T}{\partial z} \sim \frac{(1-z^2)^{3/2}(1-\rho^2)^{\frac{1}{2}}(1-2rz+z^2)^{\frac{n}{2}-2}}{(1-2\rho r+\rho^2-2u)^{\frac{n}{2}-2}\{(1-\rho z)[1-\rho r-(r-\rho)z]-u(1-z^2)\}} \quad .$$

Hence

$$\frac{\partial}{\partial u}\left[M(u-rT,T)\frac{\partial T}{\partial z}\right] \sim \frac{(1-z^2)^{3/2}(1-\rho^2)^{\frac{1}{2}}(1-2rz+z^2)^{\frac{n}{2}-2}}{(1-2\rho r+\rho^2-2u)^{\frac{n}{2}-1}\{(1-\rho z)[1-\rho r-(r-\rho)z]-u(1-z^2)\}}$$

$$\times \left\{ n-4+\frac{(1-z^2)(1-2\rho r+\rho^2-2u)}{\{(1-\rho z)[1-\rho r-(r-\rho)z]-u(1-z^2)\}} \right\}$$

so that

$$\frac{\partial}{\partial u}\left[M(u-rT,T)\frac{\partial T}{\partial z}\right]\Bigg|_{u=0}$$

$$\sim \frac{n(1-z^2)^{3/2}(1-\rho^2)^{\frac{1}{2}}(1-2rz+z^2)^{\frac{n}{2}-2}}{(1-2\rho r+\rho^2)^{\frac{n}{2}-1}\,(1-\rho z)[1-\rho r-(r-\rho)z]}$$

$$* \left\{1+\frac{1}{n}\left[\frac{(1-z^2)(1-2\rho r+\rho^2)}{(1-\rho z)[1-\rho r-(r-\rho)z]}-4\right]\right\} \quad .$$

Thus equation (I.2) gives us

$$\text{(2.2.26)} \qquad h(r) \sim \frac{n(1-\rho^2)^{\frac{1}{2}}}{2\pi i(1-2\rho r+\rho^2)^{\frac{n}{2}-1}} \int \varphi(z)(1-2rz+z^2)^{\frac{n}{2}-2}\, dz \ ,$$

where

$$\text{(2.2.27)} \quad \varphi(z) = \frac{(1-z^2)^{3/2}}{(1-\rho z)[1-\rho r-(r-\rho)z]}\left\{1+\frac{1}{n}\left[\frac{(1-z^2)(1-2\rho r+\rho^2)}{(1-\rho z)[1-\rho r-(r-\rho)z]}-4\right]\right\} \quad .$$

The subsequent discussion in this section is essentially that given by Patton [25].

With $u = 0$, the mapping $z + \frac{1}{z} = \frac{1+\rho^2-2rT}{\rho+T}$ is easily seen to consist of the following elementary mappings:

$$\text{(2.2.28)} \quad \begin{cases} \text{A:} & V = \rho + T \\ \text{B:} & U = \dfrac{V}{1-2\rho r+\rho^2} \\ \text{C:} & S = \dfrac{1}{2U} \\ \text{D:} & Q = S + r \\ \text{E:} & z + \dfrac{1}{z} = 2Q \end{cases} \quad .$$

A maps the T-plane cut along the parts of the real axis exterior to the interval $\left\{ \frac{-(1+\rho)^2}{2(1+r)} , \frac{(1-\rho)^2}{2(1-r)} \right\}$ onto the V-plane cut along the parts of the real axis exterior to the interval $\left\{ \frac{-(1-2\rho r+\rho^2)}{2(1+r)} , \frac{1-2\rho r+\rho^2}{2(1-r)} \right\}$. B maps the region of the V-plane, as above, onto the U-plane cut along the real axis exterior to the interval $\left\{ \frac{-1}{2(1+r)} , \frac{1}{2(1-r)} \right\}$. C maps this region of the U-plane onto the S-plane cut along the real axis from $-(1+r)$ to $(1-r)$. This region of the S-plane is mapped onto the Q-plane cut along the real axis from -1 to 1 by D, and this region of the Q plane is mapped onto the interior of the unit circle, $|z| = 1$, in the z plane by E. Hence the net effect of the transformation is to map the whole T-plane cut along the real axis exterior to the interval $\left\{ \frac{-(1+\rho)^2}{2(1+r)} , \frac{(1-\rho)^2}{2(1-r)} \right\}$ onto the interior of the unit circle.

Since by equation (2.2.19)

$$z + \frac{1}{z}\bigg|_{u=o} = \frac{1 + \rho^2 - 2\rho r}{\rho + T} + 2r \quad ,$$

as $T \rightarrow \pm\, i\,\infty$, $z + \frac{1}{z} \rightarrow 2r$.

If

$$z + \frac{1}{z} = 2r$$

then

$$z^2 - 2rz + 1 = 0$$

and

$$\begin{aligned} z &= r \pm i\,\sqrt{1 - r^2} \\ &= e^{\pm i\theta} \end{aligned}$$

where

$$r = \cos\theta \ .$$

We also note that if $T = 0$

$$z + \frac{1}{z} = \rho + \frac{1}{\rho}$$

and hence, the transformed path in the z-plane cuts the real axis at $z = \rho$. Thus, the path of integration in the z-plane crosses the real axis at $z = \rho$ and ends on the boundary of the unit circle at $e^{-i\theta}$ and $e^{i\theta}$ where $r = \cos\theta$.

The only possible singularity in the integrand of the integral for $h(r)$ would be the real singularity arising from

$$1 - \rho r - (r-\rho)z = 0 \ .$$

This singularity is avoided if, on the real axis, we confine z to the following range:

(i) $\qquad \rho \leq z \leq r , \qquad \rho < r \ .$

Now, we may write

$$1 - \rho r - (r-\rho)z = (1-z)(r-\rho) + (1-r)(1+\rho) \ .$$

Then, since $\qquad |\rho| < 1$ and $|r| < 1$,

we see that

$$1 - z > 0, \quad r - \rho > 0, \quad 1 - r > 0 \quad \text{and} \quad 1 + \rho > 0 \ .$$

From this it can be seen that

$$(1-z)(1-\rho) > 0 \quad \text{and} \quad (1-r)(1+\rho) > 0 \ ,$$

and finally that

$$1 - \rho r - (r-\rho)z > 0 \quad .$$

(ii) $\quad r \leq z \leq \rho \,, \quad r < \rho \quad .$

Here,

$$1 - \rho r - (r-\rho)z = 1-r^2 + (\rho-r)(z-r) \quad .$$

As

$$|\rho| < 1 \,, \quad \text{and} \quad |r| < 1 \quad ,$$

we have

$$1 - r^2 > 0 \,, \quad \rho - r > 0 \quad \text{and} \quad z - r \geq 0 \quad .$$

Then

$$(\rho-r)(z-r) \geq 0$$

and hence

$$1 - \rho r - (r-\rho)z > 0$$

(iii) $\quad z = r = \rho \quad .$

Since $|\rho| < 1$ and $|r| < 1$,
$1 - \rho r > 0$ and $(r-\rho)z = 0$

so that

$$1 - \rho r - (r-\rho)z > 0 \quad .$$

Since there is no singularity in the closed interval joining ρ and r, we may deform the path of integration in the z-plane to be the straight line joining $e^{-i\theta}$ to $e^{i\theta}$ and crossing the real axis at $z = r$.

On the path of integration we may write

$$z = r + i\,\omega\,(1-r^2)^{\frac{1}{2}} \tag{2.2.29}$$

where ω is the real variable with

$$-1 \leq \omega \leq 1 \quad .$$

Then

$$1 - 2rz + z^2 = (1-r^2)(1-\omega^2) \tag{2.2.30}$$

and

$$dz = i(1-r^2)^{\frac{1}{2}}\, d\omega \quad . \tag{2.2.31}$$

Substituting equation (2.2.29) in equation (2.2.26), we have

$$h(r) \sim \frac{n(1-\rho^2)^{\frac{1}{2}}\,(1-r^2)^{\frac{n-3}{2}}}{2\pi(1-2\rho r+\rho^2)^{\frac{n}{2}-1}} \int_{-1}^{1} \varphi(z)\,(1-\omega^2)^{\frac{n}{2}-2}\, d\omega \quad . \tag{2.2.32}$$

The integrand of $h(r)$ is of the form

$$\varphi(z)\,[\psi(z)]^{\frac{n}{2}-2}$$

where

$$\psi(z) = 1 - 2rz + z^2 \quad .$$

Since the integral cannot be readily evaluated in closed form, we expand $\varphi(z)$ as a power series in $z - r$, where

$$\psi'(r) = 0 \quad ,$$

and integrate with respect to ω. Using equations (2.2.29) we have

$$z - r = i\,\omega\,(1-r^2)^{\frac{1}{2}}$$

and hence,

$$\varphi(z) = \varphi(r) + \varphi'(r)(z-r) + \ldots + \frac{\varphi^{(k)}(r)\,(z-r)^k}{k\,!} + \ldots$$

or

$$\varphi(z) = \varphi(r) + \varphi'(r)\,i\,\omega(1-r^2)^{\frac{1}{2}} + \ldots + \frac{i^k\omega^k(1-r^2)^{\frac{k}{2}}}{k!}\,\varphi^{(k)}(r) + \ldots \quad . \tag{2.2.33}$$

Substituting this equation in (2.2.32), we get

$$h(r) \sim \frac{n(1-\rho^2)^{\frac{1}{2}}(1-r^2)^{\frac{n-3}{2}}}{2\pi(1-2\rho r+\rho^2)^{\frac{n}{2}-1}} \sum_{k=0}^{\infty} \frac{i^k(1-r^2)^{\frac{k}{2}}\,\varphi^{(k)}(r)}{k!} \int_{-1}^{1} \omega^k(1-\omega^2)^{\frac{n}{2}-2}\,d\omega \;. \tag{2.2.34}$$

Now, if

$$k = 2m + 1 \;, \quad m = 0,1,2,\ldots$$

then

$$\omega^k\,(1-\omega^2)^{\frac{n}{2}-2}$$

is an odd function and the integral vanishes. Therefore, equation (2.2.34) reduces to

$$h(r) \sim \frac{n(1-\rho^2)^{\frac{1}{2}}(1-r^2)^{\frac{n-3}{2}}}{2\pi(1-2\rho r+\rho^2)^{\frac{n}{2}-1}} \sum_{k=0}^{\infty} \frac{(-1)^k(1-r^2)^k\,\varphi^{(2k)}(r)}{(2k)!} \tag{2.2.35}$$

$$\times \int_{-1}^{1} \omega^{2k}(1-\omega^2)^{\frac{n}{2}-2}\,d\omega \;.$$

To evaluate

$$I_k = \int_{-1}^{1} \omega^{2k} (1-\omega^2)^{\frac{n}{2}-2} \, d\omega \tag{2.2.36}$$

we proceed in the following manner: Taking

$$\omega^2 = u, \quad d\omega = \tfrac{1}{2} u^{-\frac{1}{2}} du \quad ,$$

we have

$$I_k = \int_0^1 u^{k-\frac{1}{2}} (1-u)^{\frac{n}{2}-2} \, du$$

$$= \frac{\Gamma(k+\frac{1}{2})\,\Gamma(\frac{n}{2}-1)}{\Gamma(k+\frac{n}{2}-\frac{1}{2})} \quad .$$

Thus

$$I_o = \frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{n}{2}-\frac{1}{2})}$$

$$= \frac{\sqrt{\pi}\,\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{n-1}{2})}$$

and

$$I_k = \frac{1.3.5. \cdots (2k-1)}{(n-1)(n+1)\cdots(n+2k-3)} \; I_o \quad . \tag{2.2.37}$$

Substituting equation (2.2.37) in equation (2.2.35) we get

$$h(r) \sim \frac{n(1-\rho^2)^{\frac{1}{2}}(1-r^2)^{\frac{n-3}{2}} I_o}{2\pi(1-2\rho r+\rho^2)^{\frac{n}{2}-1}}$$

$$\times \left\{\varphi(r) + \sum_{k=1}^{\infty} \frac{(-1)^k(1-r^2)^k}{(2k)!} \varphi^{(2k)}(r) \frac{[1.3. \cdots (2k-1)]}{[(n-1)(n+1) \cdots (n+2k-3)]}\right\}$$

or

$$h(r) \sim \frac{n\,\Gamma(\frac{n}{2}-1)(1-\rho^2)^{\frac{1}{2}}(1-r^2)^{\frac{n}{2}-1}}{2\sqrt{\pi}\,\Gamma(\frac{n-1}{2})(1-\rho r)(1-2\rho r+\rho^2)^{\frac{n}{2}-1}} \{1 + O(n^{-1})\} \tag{2.2.38}$$

since

$$\varphi(r) = \frac{(1-r^2)^{\frac{1}{2}}}{1-\rho r}\left\{1 + \frac{1}{n}\left[\frac{1-2\rho r+\rho^2}{1-\rho r} - 4\right]\right\}$$

$$= \frac{(1-r^2)^{\frac{1}{2}}}{1-\rho r}\{1 + O(n^{-1})\} .$$

Patton ([25] in Chapter II Section 4) has shown the following:

(i) $$\frac{\Gamma(\frac{n}{2} - 1)}{\Gamma(\frac{n-1}{2})} \quad \text{is} \quad O(n^{-\frac{1}{2}}) \quad ,$$

(ii) $$h(\rho) \sim \left[\frac{n}{2\pi(1 - \rho^2)} \right]^{\frac{1}{2}} \quad \{1 + O(n^{-1})\} \quad ,$$

(iii) $$h(r) \quad \text{is} \quad O(n^{\frac{1}{2}}) \quad \text{at} \quad r = \rho \quad ,$$

(iv) Var (r) is $O(n^{-1})$ which suggests that the distribution of r is concentrated about $r = \rho$ within a range which is $O(n^{-\frac{1}{2}})$ on either side of ρ .

(v) Over the effective range of r, $(r-\rho)$ can be taken as $O(n^{-\frac{1}{2}})$.

(vi) If r is replaced by ρ in the first neglected term in the expansion of $h(r)$ the term is altered by an amount $O(n^{-3/2})$.

(vii) $$\frac{(1 - \rho^2)^{\frac{1}{2}}(1 - r^2)^{\frac{1}{2}}}{1 - \rho r} = \left(\frac{1-r^2}{1-2\rho r+\rho^2}\right)^{\frac{1}{2(1-\rho^2)}} [1 + O(n^{-3/2})]$$

(viii) $$h(r) \sim K \frac{(1-r^2)^{\frac{N-1}{2}}}{(1-2\rho r+\rho^2)^{\frac{N}{2}}} \quad [1 + O(n^{-3/2})]$$

where k is an adjusted normalising constant and

$$N = n - 1 + \frac{\rho^2}{1 - \rho^2} \quad .$$

Below we describe the renormalization of $h(r)$ as detailed by Patton [25] Chapter II Section 4.

Consider $$J_k = \int_{-1}^{1} \frac{(1-r^2)^{\frac{k-1}{2}}}{(1-2\rho r+\rho^2)^{\frac{k}{2}}}\, dr \qquad k = 1,2,\ldots$$

and let

$$u = \frac{1+r}{2} \quad .$$

Then

$$J_k = \frac{2^k}{(1+\rho)^k} \int_0^1 u^{\frac{k-1}{2}} (1-u)^{\frac{k-1}{2}} \left[1 - \frac{4\rho u}{(1+\rho)^2}\right]^{-\frac{k}{2}} du$$

It can be shown (see [41], page 293, example No. 1) that

$$J_k = \frac{2^k}{(1+\rho)^k} \frac{[\Gamma(\frac{k+1}{2})]^2}{\Gamma(k+1)} F\left[\frac{k}{2}, \frac{k+1}{2}; k+1; \frac{4\rho}{(1+\rho)^2}\right] \tag{2.2.39}$$

where $F(a,b;c;z)$ is the usual hypergeometric function. It follows from ([10], page 64, formula 24) and the hypergoemetric power series that

$$F\left[\frac{k}{2}, \frac{k+1}{2}; k+1; \frac{4\rho}{(1+\rho)^2}\right] = (1+\rho)^k F\left[\frac{k}{2}, 0; \frac{k}{2}+1; \rho^2\right]$$

$$= (1+\rho)^k \quad .$$

Then equation (2.2.39) becomes

$$J_k = \frac{2^k [\Gamma(\frac{k+1}{2})]^2}{\Gamma(k+1)} \quad .$$

Using the duplication formula for the gamma function, ([41], page 240) we have

$$J_k = \frac{2^k \, [\Gamma(\frac{k+1}{2})]^2}{2^k \, (\pi)^{-\frac{1}{2}} \, \Gamma(\frac{k+1}{2}) \, \Gamma(\frac{k}{2}+1)}$$

$$= \frac{\sqrt{\pi} \, \Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2}+1)} \quad .$$

Thus for $k = 1,2,\ldots$ we have

$$J_k = \int_{-1}^{1} \frac{(1-r^2)^{\frac{k-1}{2}}}{(1-2\rho r+\rho^2)^{\frac{k}{2}}} \, dr = \frac{\sqrt{\pi} \, \Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2}+1)} \quad . \tag{2.2.40}$$

Employing this general result, we see that

$$I = \int_{-1}^{1} \frac{(1-r^2)^{\frac{N-1}{2}}}{(1-2\rho r+\rho^2)^{\frac{N}{2}}} \, dr$$

$$= \frac{\sqrt{\pi} \, \Gamma(\frac{N+1}{2})}{\Gamma(\frac{N}{2}+1)} \quad .$$

Then the renormalized density function is given by

$$h(r) \sim K(n,\rho) \, \frac{(1-r^2)^{\frac{N-1}{2}}}{(1-2\rho r+\rho^2)^{\frac{N}{2}}} \, [1 + O(n^{-3/2})] \tag{2.2.41}$$

where

$$K(n,\rho) = \frac{1}{I} = \frac{\Gamma(\frac{N}{2}+1)}{\sqrt{\pi}\,\Gamma(\frac{N+1}{2})} \tag{2.2.42}$$

and

$$N = n - 1 + \frac{\rho^2}{1 - \rho^2} .$$

This is Leipnik's [20] form of $h(r)$.

Daniels [5] also considered the case of the Unknown Mean. Below we summarize his results.

When the mean is unknown, it is estimated by

$$\bar{x} = \frac{\frac{1}{2}x_1 + x_2 + \ldots + x_{n-1} + \frac{1}{2}x_n}{n-1} .$$

The serial correlation coefficient, ρ , is estimated by

$$r = \frac{C}{C_o}$$

where

$$C = (x_1-\bar{x})(x_2-\bar{x})+\ldots+(x_{n-1}-\bar{x})(x_n-\bar{x}) = c-(n-1)\bar{x}^2$$

and $$C_o = \tfrac{1}{2}(x_1-\bar{x})^2+(x_2-\bar{x})^2+\ldots+(x_{n-1}-\bar{x})^2+\tfrac{1}{2}(x_n-\bar{x})^2 = c_o-(n-1)\bar{x}^2$$

with c, c_o defined as in equations (2.2.6) and (2.2.7).

The joint moment-generating function of C and C_o is

$$M(T_o,T) = \mathcal{E}(e^{T_oC_o + TC})$$

$$= \frac{(1-\rho^2)^{\frac{1}{2}}}{(2\pi)^{\frac{n}{2}}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{T_oC_o+TC-\frac{1}{2}\underline{x}'\underline{C}^{-1}\underline{x}} dx_1 \ldots dx_n$$

where $\underline{C}^{-1}$ is given by equation (2.2.3). Since

$$T_oC_o+TC-\tfrac{1}{2}\underline{x}'\underline{C}^{-1}\underline{x} = T_oc_o+Tc-\tfrac{1}{2}\underline{x}'\underline{C}^{-1}\underline{x}-(n-1)(T_o+T)\bar{x}^2$$

$$= -\tfrac{1}{2}\,\underline{x}'\underline{B}\,\underline{x} - \frac{(T_o+T)}{n-1}\,\underline{x}'\,\underline{m}\,\underline{m}'\,\underline{x}$$

$$= -\tfrac{1}{2}\,\underline{x}'\left[\underline{B} + \frac{2(T_o+T)}{n-1}\,\underline{m}\,\underline{m}'\right]\underline{x}$$

with $\underline{B}$ given by equation (2.2.8),

$$\underline{x}' = (x_1, x_2, \ldots, x_n)$$

and

$$\underline{m}' = (\tfrac{1}{2}, 1, 1, \ldots, 1, 1, \tfrac{1}{2}) \quad ,$$

we have

$$M(T_o,T) = \frac{(1-\rho^2)^{\frac{1}{2}}}{(2\pi)^{\frac{n}{2}}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\frac{1}{2}\underline{x}'\left[\underline{B}+\frac{2(T_o+T)}{n-1}\underline{m}\,\underline{m}'\right]\underline{x}}$$

$$dx_1 dx_2 \ldots dx_n \quad ,$$

which gives us

$$M(T_o,T) = (1-\rho^2)^{\frac{1}{2}} \left|\underline{B} + \frac{2(T_o+T)}{n-1} \underline{m}\,\underline{m}'\right|^{-\frac{1}{2}} .$$

On evaluating $\left|\underline{B} + \frac{2(T_o+T)}{n-1} \underline{m}\,\underline{m}'\right|$ and making the necessary substitutions and reductions, we obtain the following approximation after omitting a term in z^{n-1}:

$$M(u-rT,T) \sim \frac{(1-\rho)^{\frac{1}{2}}(1-z^2)^{\frac{1}{2}}(1-z)(1-2rz+z^2)^{\frac{n-1}{2}}}{(1-\rho)\{(1-\rho z)[1-\rho r-(r-\rho)z]-u(1-z^2)\}(1-2\rho r+\rho^2-2u)^{\frac{n-3}{2}}}$$

$$\times\left\{1 + \frac{(1+\rho)^2(1-2rz+z^2)[(z-\rho)(1-\rho z)(1-r)+u(1-z^2)]}{(n-1)(1-z^2)(1-2\rho r+\rho^2-2u)[1-\rho r-(r-\rho)z-u(1-z^2)]}\right\}$$

with $$z + \frac{1}{z} = \frac{1+\rho^2-2T_o}{\rho+T} .$$

To obtain the approximation with remainder relatively $O(n^{-3/2})$, we ignore the last factor and we take the dominant term as before in the expansion of the integral for $h(r)$. On renormalizing, we have

$$h(r) \sim \frac{K(1-\rho^2)^{\frac{1}{2}}(1-r^2)^{\frac{n-3}{2}}(1-r)}{(1-\rho r)(1-2\rho r+\rho^2)^{\frac{n-3}{2}}} \quad [1 + O(n^{-3/2})]$$

where K is the normalizing constant. In Leipnik's form,

$$(2.2.43)\quad h(r) \sim \frac{2}{\sqrt{\pi}} \frac{\Gamma\left(\frac{N+3}{2}\right)}{\Gamma\left(\frac{N}{2}\right)\ [N(1-\rho)+(1+\rho)]} \frac{(1-r)(1-r^2)^{\frac{N}{2}-1}}{(1-2\rho r+\rho^2)^{\frac{N-1}{2}}} \quad [1+O(n^{-3/2})] ,$$

with

$$N = n-1+\frac{\rho^2}{1-\rho^2} .$$

3. <u>Non-circular Markov Process; Patton's Case</u>.

Consider a non-circular Markov Process described in (2.1.1) as taken by Daniels. Take the sample estimate of ρ to be

$$r = \frac{c}{c_o}$$

where

$$(2.3.1)\qquad \begin{aligned} c &= \tfrac{1}{2} x_1^2 + x_1 x_2 + \dots + x_{n-1} x_n + \tfrac{1}{2} x_n^2 \\ c_o &= x_1^2 + x_2^2 + \dots + x_n^2 \end{aligned}$$

Following Daniels' Method, Patton [25] obtained the following:

The joint moment-generating function of c and c_o is

$$M(T_o,T) = \mathcal{E}\left(e^{T_o c_o + Tc}\right)$$

$$= \frac{(1-\rho^2)^{\frac{1}{2}}}{(2\pi)^{\frac{n}{2}}} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} e^{T_o c_o + Tc - \frac{1}{2}\underline{x}' \underline{C}^{-1} \underline{x}} \, dx_1 \dots dx_n$$

where $\underline{C}^{-1}$ is given by the equation (2.2.3).

Since

$$T_oc_o+Tc - \tfrac{1}{2}\,\underline{x}'\underline{C}^{-1}\underline{x} = -\tfrac{1}{2}[(1-2T_o-T)x_1^2 + (1+\rho^2-2T_o)(x_2^2+\ldots+x_{n-1}^2)$$

$$+ (1-2T_o-T)x_n^2 - 2(\rho+T)(x_1x_2+\ldots+x_{n-1}x_n)]$$

$$= -\tfrac{1}{2}\,\underline{x}'\,\underline{D}\,\underline{x} \quad ,$$

where

$$(2.3.2) \qquad \underline{D} = \begin{bmatrix} 1-2T_o-T & -(\rho+T) & & & & & \\ -(\rho+T) & 1+\rho^2-2T_o & -(\rho+T) & & & (0) & \\ & -(\rho+T) & 1+\rho^2-2T_o & -(\rho+T) & & & \\ & & & \cdots\cdots\cdots\cdots & & & \\ & & & & -(\rho+T) & 1+\rho^2-2T_o & -(\rho+T) \\ & (0) & & & & -(\rho+T) & 1-2T_o-T \end{bmatrix}$$

and

$$\underline{x}' = (x_1, \ldots, x_n)$$

it is seen that

$$(2.3.3) \qquad M(T_o,T) = \frac{(1-\rho^2)^{\frac{1}{2}}}{(2\pi)^{\frac{n}{2}}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\frac{1}{2}\,\underline{x}'\underline{D}\,\underline{x}} \, dx_1dx_2\ldots dx_n \quad .$$

Diagonalizing $\underline{D}$ and integrating, we obtain

$$(2.3.4) \qquad M(T_o,T) = (1-\rho^2)^{\frac{1}{2}} \; |\underline{D}|^{-\frac{1}{2}}$$

where $|\underline{D}|^{-\frac{1}{2}}$ is the Jacobian of the transformation.

On evaluating $|\underline{D}|$ and making necessary substitutions and simplifications it is found that

$$(2.3.5) \qquad |\underline{D}| \sim \frac{(1-2\rho r+\rho^2-2u)^{n-2}}{(1-z^2)(1-2rz+z^2)^n} \; [(1-\rho z)(1-2\rho r+\rho z+\rho-z)-2u(1-z)]^2$$

and therefore

$$M(u-rT,T) \sim \frac{(1-\rho^2)^{\frac{1}{2}}(1-z^2)^{\frac{1}{2}}(1-2rz+z^2)^{\frac{n}{2}}}{(1-2\rho r+\rho^2-2u)^{\frac{n}{2}-1} \; [(1-\rho z)(1-2\rho r+\rho z+\rho-z)-2u(1-z)]} .$$

Then using (I.2) we obtain

$$(2.3.6) \qquad h(r) \sim \frac{n(1-\rho^2)^{\frac{1}{2}}}{2\pi i(1-2\rho r+\rho^2)^{\frac{n}{2}-1}} \int \varphi(z) \; (1-2rz+z^2)^{\frac{n}{2}-2} \; dz$$

where

$$(2.3.7) \qquad \varphi(z) = \frac{(1-z^2)^{3/2}}{(1-\rho z)(1-2\rho r+\rho z+\rho-z)} \left\{1 + \frac{1}{n}\left[\frac{2(1-z)(1-2\rho r+\rho^2)}{(1-\rho z)(1-2\rho r+\rho z+\rho-z)} - 4\right]\right\} .$$

To obtain an approximation with remainder relatively $O(n^{-3/2})$ the last factor was ignored and the dominant term was taken in the expansion of the integral for $h(r)$.

On renormalizing $h(r)$ was obtained as

(2.3.8) $$h(r) \sim K(n,\rho) \frac{(1-r^2)^{\frac{N+1}{2}}}{(1-r)(1-2\rho r+\rho^2)^{\frac{N}{2}}} \quad [1 + 0(n^{-3/2})]$$

where

(2.3.9) $$K(n,\rho) = \frac{2\,\Gamma(\frac{N}{2} + 2)}{\sqrt{\pi}\,\Gamma(\frac{N+1}{2})\,[N(1+\rho) + 2]}$$

and

(2.3.10) $$N = n - 1 + \frac{\rho^2}{1 - \rho^2}$$

which is in Leipnik's Form.

If the mean is unknown we take as the estimate of the unknown mean

(2.3.11) $$\bar{x} = \frac{x_1 + \dots + x_n}{n}$$

and the estimate r of the serial correlation coefficient, ρ, to be

(2.3.12) $$r = \frac{C}{C_o}$$

where

(2.3.13) $$\left\{\begin{aligned} C &= \tfrac{1}{2}(x-\bar{x})^2 + (x_1-\bar{x})(x_2-\bar{x}) + \dots + (x_{n-1}-\bar{x})(x_n-\bar{x}) + \tfrac{1}{2}(x_n-\bar{x})^2 \\ &= c - n\,\bar{x}^2 , \\ C_o &= (x_1-\bar{x})^2 + \dots + (x_n-\bar{x})^2 \\ &= c_o - n\,\bar{x}^2 \end{aligned}\right.$$

with c and c_o as defined in(2.3.1). Using Daniels' method it is found that

$$(2.3.14) \qquad h(r) \sim \frac{K\,(1-r^2)^{\frac{N}{2}}}{(1-2\rho r+\rho^2)^{\frac{N-1}{2}}} \quad [1 + O(n^{-3/2})]$$

where K is an adjusted normalizing constant and

$$N = n - 1 + \frac{\rho^2}{1-\rho^2} \quad .$$

In this case, the renormalized density function, $h(r)$, in Leipnik's form is obtained as

$$(2.3.15) \qquad h(r) \sim K(n,\rho) \; \frac{(1-r^2)^{\frac{N}{2}}}{(1-2\rho r+\rho^2)^{\frac{N-1}{2}}} \quad [1 + O(n^{-3/2})]$$

where

$$(2.3.16) \qquad K(n,\rho) = \frac{2}{\sqrt{\pi}} \; \frac{\Gamma(\frac{N+5}{2})}{\Gamma(\frac{N}{2}+1)\,[(1-\rho^2)(N-1)+4]}$$

and

$$N = n - 1 + \frac{\rho^2}{1-\rho^2} \quad .$$

CHAPTER III

THE APPROXIMATE DISTRIBUTION OF ESTIMATES OF THE SERIAL CORRELATION COEFFICIENT FOR A PRESTATIONARY LINEAR MARKOV PROCESS WITH KNOWN MEAN

In the process defined by (2.1.2) following Daniels [5] we take the sample estimate of ρ to be

$$r = \frac{c}{c_o}$$

where

$$(3.1.1) \qquad \begin{cases} c = x_1 x_2 + x_2 x_3 + \dots + x_{n-1}\, x_n \ , \\ c_o = \frac{1}{2}\, x_1^2 + x_2^2 + x_3^2 + \dots + x_{n-1}^2 + \frac{1}{2}\, x_n^2 \ , \end{cases}$$

and employ a method similar to that used by Daniels for obtaining the approximate distribution of r.

In matrix notation (2.1.2) takes the form

$$\underline{x} = \underline{R}\ \underline{e}$$

where

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \underline{R} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ \rho & 1 & 0 & \dots & 0 & 0 \\ \rho^2 & \rho & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \rho^{n-1} & \rho^{n-2} & \rho^{n-3} & \dots & \rho & 1 \end{bmatrix} \quad \text{and} \quad \underline{e} = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}.$$

Hence $\underline{e} = \underline{R}^{-1}\underline{x}$.

We can also write (2.1.2) in the following form

$$
\begin{aligned}
e_1 &= x_1 \\
e_2 &= x_2 - \rho x_1 \\
e_3 &= x_3 - \rho x_2 \\
&\cdots\cdots\cdots \\
e_n &= x_n - \rho x_{n-1}
\end{aligned}
$$

which gives us

$$
\underline{e} = \begin{bmatrix} 1 & & & & & (0) \\ -\rho & 1 & & & & \\ & -\rho & 1 & & & \\ & & & \cdots\cdots & & \\ & (0) & & -\rho & 1 & \\ & & & & -\rho & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} .
$$

Hence

$$
\underline{R}^{-1} = \begin{bmatrix} 1 & & & & & (0) \\ -\rho & 1 & & & & \\ & -\rho & 1 & & & \\ & & & \cdots\cdots & & \\ & (0) & & -\rho & 1 & \\ & & & & -\rho & 1 \end{bmatrix}
$$

and

$$(\underline{R}^{-1})' = \begin{bmatrix} 1 & -\rho & & & & \\ & 1 & -\rho & & (0) & \\ & & 1 & -\rho & & \\ & & & \cdots\cdots & & \\ & (0) & & & 1 & -\rho \\ & & & & & 1 \end{bmatrix}.$$

Thus $\underline{C}^{-1} = (R^{-1})' \, \underline{R}^{-1}$

$$(3.1.2) \quad = \begin{bmatrix} 1+\rho^2 & -\rho & & & & & \\ -\rho & 1+\rho^2 & -\rho & & & (0) & \\ & -\rho & 1+\rho^2 & -\rho & & & \\ & & & \cdots\cdots & & & \\ & (0) & & & -\rho & 1+\rho^2 & -\rho \\ & & & & & -\rho & 1 \end{bmatrix}.$$

Hence

$$\underline{x}' \, \underline{C}^{-1} \, \underline{x} = x' \, (\underline{R}^{-1})' \, (\underline{R}^{-1}) \, \underline{x}$$

$$= [x_1(1+\rho^2) - \rho x_2, \; -\rho x_1 + (1+\rho^2)x_2 - \rho x_3 \, ,$$

$$\ldots, \; -\rho x_{n-2} + (1+\rho^2)x_{n-1} - \rho x_n, \; -\rho x_{n-1} + x_n] \quad \times \quad \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$$

$$
\begin{aligned}
= \quad & x_1^2(1+\rho^2) - \rho\, x_1x_2 \\
& + x_2^2(1+\rho^2) - \rho\, x_1x_2 - \rho\, x_2x_3 \\
& + x_3^2(1+\rho^2) \qquad\qquad - \rho\, x_2x_3 - \rho\, x_3x_4 \\
& + x_4^2(1+\rho^2) - \rho\, x_4x_5 \qquad\qquad - \rho\, x_3x_4 \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& + x_{n-2}^2(1+\rho^2) - \rho\, x_{n-3}x_{n-2} - \rho\, x_{n-2}x_{n-1} \\
& + x_{n-1}^2(1+\rho^2) \qquad\qquad - \rho\, x_{n-2}x_{n-1} - \rho\, x_{n-1}x_n \\
& + x_n^2 \qquad\qquad\qquad\qquad - \rho\, x_{n-1}x_n \\
= \quad & (1+\rho^2) \sum_{i=1}^{n-1} x_i^2 - 2\rho \sum_{i=1}^{n-1} x_i x_{i+1} + x_n^2 \ .
\end{aligned}
$$

The joint distribution of $x_1, x_2, \ldots, x_n$ is given by

$$
dF(x_1,x_2,\ldots,x_n) = (2\pi)^{-\frac{n}{2}} \ |\underline{C}|^{-\frac{1}{2}} \exp(-\tfrac{1}{2}\, \underline{x}'\ \underline{C}^{-1}\ \underline{x})\ dx_1dx_2\ldots dx_n \ .
$$

Now

$$
\begin{aligned}
|\underline{C}^{-1}| &= |(\underline{R}^{-1})'\ \underline{R}^{-1}| \\
&= |\underline{R}^{-1}|^2 \\
&= 1 \ .
\end{aligned}
$$

Thus

$$dF(x_1,x_2,\ldots,x_n) = (2\pi)^{-\frac{n}{2}} \exp(-\tfrac{1}{2}\,\underline{x}'\,\underline{C}^{-1}\,\underline{x})\ \ dx_1 dx_2 \ldots dx_n \quad .$$

Then the joint moment-generating function of c and c_o is

$$M(T_o,T) = \mathcal{E}[\exp(T_o c_o + Tc)]$$

$$= (2\pi)^{-\frac{n}{2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{T_o c_o + Tc - \frac{1}{2}\underline{x}'\underline{C}^{-1}\,\underline{x}}\ \ dx_1 dx_2 \ldots dx_n$$

where $\underline{C}^{-1}$ is given by (3.1.2). Now,

$$\begin{aligned} T_o c_o + Tc - \tfrac{1}{2}\underline{x}'\underline{C}^{-1}\,\underline{x} &= -\tfrac{1}{2}\Big\{(1+\rho^2)\sum_{i=1}^{n-1} x_i^2 - 2\rho\sum_{i=1}^{n-1} x_i x_{i+1} + x_n^2 \\ &\qquad - 2T_o(\tfrac{1}{2}x_1^2 + x_2^2 + \ldots + x_{n-1}^2 + \tfrac{1}{2}x_n^2) \\ &\qquad - 2T(x_1x_2 + x_2x_3 + \ldots + x_{n-1}x_n)\Big\} \\ &= -\tfrac{1}{2}\Big\{(1+\rho^2-T_o)x_1^2 + (1+\rho^2-2T_o)(x_2^2+\ldots+x_{n-1}^2) + (1-T_o)x_n^2 \\ &\qquad - 2(\rho+T)(x_1x_2 + x_2x_3 + \ldots + x_{n-1}x_n)\Big\} \\ &= -\tfrac{1}{2}\,\underline{x}'\ \underline{B}\ \underline{x} \end{aligned}$$

where

$$\underline{B} = \begin{bmatrix} 1+\rho^2-T_o & -(\rho+T) & & & & & \\ -(\rho+T) & 1+\rho^2-2T_o & -(\rho+T) & & & (0) & \\ & -(\rho+T) & 1+\rho^2-2T_o & -(\rho+T) & & & \\ & & & \cdots\cdots\cdots & & & \\ & & & & -(\rho+T) & 1+\rho^2-2T_o & -(\rho+T) \\ & (0) & & & & -(\rho+T) & 1-T_o \end{bmatrix}_n . \tag{3.1.3}$$

Then

$$M(T_o,T) = (2\pi)^{-\frac{n}{2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\frac{1}{2}\underline{x}'\underline{B}\,\underline{x}} \; dx_1 \ldots dx_n \quad .$$

As mentioned in Chapter II, Section 2 , there is an orthogonal transformation of $x_1, x_2, \ldots, x_n$ that will diagonalize $\underline{B}$. Diagonalizing $\underline{B}$ and carrying out the integration, we obtain

$$M(T_o,T) = |\underline{B}|^{-\frac{1}{2}} \tag{3.1.4}$$

where $|\underline{B}|^{-\frac{1}{2}}$ is the Jacobian of the transformation.

To evaluate $|\underline{B}|$ we proceed as follows: $(\rho+T)$ is factored from each row of $|\underline{B}|$ and the following substitutions are made:

$$\begin{cases} a = \dfrac{1}{\rho + T} \\[2ex] b = \dfrac{1 + \rho^2 - 2T_o}{\rho + T} \end{cases} \tag{3.1.5}$$

so that $|\underline{B}|$ may be written as

(3.1.6) $$|\underline{B}| = (\rho + T)^n \; D_n$$

where D_n is the determinant

$$D_n = \begin{vmatrix} b+aT_o & -1 & & & & & & \\ -1 & b & -1 & & & & (0) & \\ & -1 & b & -1 & & & & \\ & & -1 & b & -1 & & & \\ & & & & \cdots\cdots\cdots & & & \\ & & & & & -1 & b & -1 \\ & (0) & & & & & -1 & a(1-T_o) \end{vmatrix}_n$$

$$= \begin{vmatrix} b & -1 & & & & & \\ -1 & b & -1 & & & (0) & \\ & -1 & b & -1 & & & \\ & & & \cdots\cdots\cdots & & & \\ & & & & -1 & b & -1 \\ & (0) & & & & -1 & a(1-T_o) \end{vmatrix}_n$$

$$+ \begin{vmatrix} aT_o & -1 & & & & & \\ & b & -1 & & & (0) & \\ & -1 & b & -1 & & & \\ & & & \cdots\cdots\cdots & & & \\ & & & & -1 & b & -1 \\ & (0) & & & & -1 & a(1-T_o) \end{vmatrix}_n .$$

Or

$$D_n = A_n + a\,T_o\,A_{n-1}\,, \tag{3.1.7}$$

where

$$A_n = \begin{vmatrix} b & -1 & & & & & (0) \\ -1 & b & -1 & & & & \\ & -1 & b & -1 & & & \\ & & -1 & b & -1 & & \\ & & & \cdots & \cdots & \cdots & \\ & & & & -1 & b & -1 \\ & (0) & & & & -1 & a(1-T_o) \end{vmatrix}_n \tag{3.1.8}$$

and A_{n-1} is obtained from A_n by removing the first row and first column.

Therefore

$$\begin{cases} A_1 = a(1 - T_o)\,, \\ A_2 = ab(1 - T_o) - 1\,, \end{cases} \tag{3.1.9}$$

and in general,

$$A_j = b\,A_{j-1} - A_{j-2}$$

or equivalently

$$A_{j+2} - b\,A_{j+1} + A_j = 0\,. \tag{3.1.10}$$

Taking E to be the forward difference operator of the calculus of finite differences defined by equation $E(A_j) = A_{j+1}$, we may write equation (3.1.10) as

$$(E^2 - bE + 1)A_j = 0 \quad .$$

Solving this difference equation, we find that the roots of the auxiliary equation

$$z^2 - bz + 1 = 0$$

are

$$z = \frac{b + \sqrt{b^2 - 4}}{2}$$

and

$$\frac{1}{z} = \frac{b - \sqrt{b^2 - 4}}{2}$$

where

(3.1.11) $$z + \frac{1}{z} = b \quad .$$

Hence the general solution of equation (3.1.10) is

(3.1.12) $$A_j = k_1 z^j + k_2 z^{-j} \quad .$$

Applying conditions (3.1.9) to equation (3.1.12), we have

$$A_1 = a(1-T_o) = k_1 z + k_2 z^{-1} \quad ,$$

$$A_2 = ab(1-T_o) - 1 = k_1 z^2 + k_2 z^{-2} \quad .$$

From these equations and equation (3.1.11) the values of k_1 and k_2 are found to be

$$k_1 = \frac{1 - a(1-T_o)z}{1 - z^2} , \qquad k_2 = \frac{\{a(1-T_o) - z\}z}{1 - z^2} .$$

Substituting these values in equation (3.1.12), and taking $j = n$ we have

$$A_n = \left\{\frac{1-a(1-T_o)z}{1 - z^2}\right\} z^n + \left[\frac{\{a(1-T_o)-z\}z}{1 - z^2}\right] z^{-n} , \tag{3.1.13}$$

or

$$A_n = \frac{z^n}{1-z^2} - \frac{z^2}{(1-z^2)z^n} - a(1-T_o)\frac{z}{1-z^2}(z^n - z^{-n}) , \tag{3.1.14}$$

where

$$a = \frac{1}{\rho + T} ,$$

$$b = \frac{1 + \rho^2 - 2T_o}{\rho + T} = z + \frac{1}{z} .$$

Substituting equations (2.2.18), (2.2.20) for u and $(\rho + T)$ respectively, we see that

$$a = \frac{1 - 2rz + z^2}{z(1 + \rho^2 - 2\rho r - 2u)} ,$$

$$1 + \rho^2 - 2T_o = (z^2 + 1)\frac{(1 + \rho^2 - 2\rho r - 2u)}{1 - 2rz + z^2} .$$

From equation (2.2.22)

$$1 - T_o = 1 - \left\{\frac{(1+z^2)(\rho r+u)-rz(1+\rho^2)}{1-2rz+z^2}\right\} \tag{3.1.15}$$

$$= \frac{1-2rz+z^2-(1+z^2)(\rho r+u)+rz(1+\rho^2)}{1-2rz+z^2} .$$

Hence

$$a(1-T_o) = \frac{1-2rz+z^2-(1+z^2)(\rho r+u)+rz(1+\rho^2)}{z(1+\rho^2-2\rho r-2u)}$$

$$= \frac{(1+z^2)(1-u-\rho r)-rz(1-\rho^2)}{z(1+\rho^2-2\rho r-2u)} \quad .$$

Then equation (3.1.14) becomes

$$A_n = \frac{z^{2n}-z^2}{z^n(1-z^2)} - \frac{(z^{2n}-1)}{(1-z^2)\ z^n} \left\{ \frac{(1+z^2)(1-u-\rho r)-rz(1-\rho^2)}{(1+\rho^2-2\rho r-2u)} \right\}$$

or

$$(3.1.16) \quad A_n = \frac{1}{z^n(1-z^2)} \left[\frac{(1-z^{2n})\{(1+z^2)(1-u-\rho r)-rz(1-\rho^2)\}}{1+\rho^2-2\rho r-2u} - z^2(1-z^{2n-2}) \right] \quad .$$

Again using equations (3.1.14) and (3.1.15) we have

$$(3.1.17) \quad aT_o\ A_{n-1} = \frac{\{(1+z^2)(\rho r+u)-rz(1+\rho^2)\}}{z(1+\rho^2-2\rho r-2u)z^{n-1}(1-z^2)} \Bigg[\frac{(1-z^{2n-2})\{(1+z^2)(1-u-\rho r)-rz(1-\rho^2)}{1+\rho^2-2\rho r-2u}$$

$$- z^2(1-z^{2n-4}) \Bigg] \quad .$$

Hence using equations (3.1.5), (3.1.6), (3.1.14) and (3.1.15) we have

$$|\underline{B}| = \frac{(1+\rho^2-2\rho r-2u)^n}{(1-2rz+z^2)^n} \frac{1}{(1-z^2)} \frac{1}{1+\rho^2-2\rho r-2u} \Bigg[(1-z^{2n})\{(1+z^2)(1-u-\rho r)-rz(1-\rho^2)\}$$

$$- z^2(1-z^{2n-4})\{(1+z^2)(\rho r+u)-rz(1+\rho^2)\} - z^2(1-z^{2n-2})(1+\rho^2-2\rho r-2u)$$

$$+ \frac{(1-z^{2n-2})\{(1+z^2)(1-u-\rho r)-rz(1-\rho^2)\}\{(1+z^2)(\rho r+u)-rz(1+\rho^2)\}}{1+\rho^2-2\rho r-2u} \Bigg] \quad .$$

We omit the terms in z^{2n} , z^{2n-2} and z^{2n-4} within the bracket which gives rise to an error that is exponentially small. Hence ignoring these terms we get

$$|\underline{B}| \sim \frac{(1+\rho^2-2\rho r-2u)^{n-1}}{(1-2rz+z^2)^n(1-z^2)}\Big[(1+z^2)(1-u-\rho r)-rz(1-\rho^2)$$

$$- z^2\{(1+z^2)(\rho r+u)-rz(1+\rho^2)\}$$

$$- z^2\ (1+\rho^2-2\rho r-2u)$$

$$+ \frac{\{(1+z^2)(1-u-\rho r)-rz(1-\rho^2)\}\{(1+z^2)(\rho r+u)-rz(1+\rho^2)\}}{1+\rho^2-2\rho r-2u}\Big]$$

$$= \frac{(1+\rho^2-2\rho r-2u)^{n-2}}{(1-2rz+z^2)^n(1-z^2)}\Big[\{(1+z^2)(1-u-\rho r)-rz(1-\rho^2)\}\{1+\rho^2-2\rho r-2u\}$$

$$- z^2\{\rho r+u+\rho rz^2+uz^2-rz-\rho^2rz+1+\rho^2-2\rho r-2u\}\{1+\rho^2-2\rho r-2u\}$$

$$+ \{(1+z^2)(1-u-\rho r)-rz(1-\rho^2)\}\{(1+z^2)(\rho r+u)-rz(1+\rho^2)\}\Big]$$

$$= \frac{(1+\rho^2-2\rho r-2u)^{n-2}}{(1-2rz+z^2)^n(1-z^2)}\Big[\{(1+z^2)(1-u-\rho r)-rz(1-\rho^2)\}$$

$$\times\{1+\rho^2-2\rho r-2u+\rho r+u+\rho rz^2+uz^2-rz-\rho^2rz\}$$

$$- z^2\{1+\rho^2-2\rho r-2u\}\{1+\rho^2-\rho r-u-rz(1+\rho^2)+z^2(\rho r+u)\}\Big]$$

$$= \frac{(1+\rho^2-2\rho r-2u)^{n-2}\{1+\rho^2-\rho r-u-rz(1+\rho^2)+z^2(\rho r+u)\}}{(1-2rz+z^2)^n\ (1-z^2)}$$

$$\times\Big[\ 1-u-\rho r-rz(1-\rho^2)+z^2(u+\rho r-\rho^2)\ \Big]$$

or

$$(3.1.18) \quad |\underline{B}| \sim \frac{(1+\rho^2-2\rho r-2u)^{n-2}\,\{1+\rho^2-\rho r-u-rz(1+\rho^2)+z^2(\rho r+u)\}}{(1-2rz+z^2)^n\,(1-z^2)}$$

$$\times \{1-u-\rho r-rz(1-\rho^2)+z^2(u+\rho r-\rho^2)\} \quad .$$

Thus an approximation of $M(T_o,T)$, as given by equation (3.1.4), is

$$(3.1.19) \quad M(u-rT,T) \sim \frac{(1-z^2)^{\frac{1}{2}}\,(1-2rz+z^2)^{\frac{n}{2}}}{(1+\rho^2-2\rho r-2u)^{\frac{n-2}{2}}\,\{1+\rho^2-\rho r-u-rz(1+\rho^2)+z^2(\rho r+u)\}^{\frac{1}{2}}}$$

$$\times \frac{1}{\{1-u-\rho r-rz(1-\rho^2)+z^2(u+\rho r-\rho^2)\}^{\frac{1}{2}}} \quad .$$

Combining equations (2.2.25) and (3.1.19) we get

$$M(u-rT,T)\,\frac{\partial T}{\partial z} \sim \frac{(1-z^2)^{3/2}\,(1-2rz+z^2)^{\frac{n}{2}-2}}{(1+\rho^2-2\rho r-2u)^{\frac{n-4}{2}}\,\{1+\rho^2-\rho r-u-rz(1+\rho^2)+z^2(\rho r+u)\}^{\frac{1}{2}}}$$

$$\times \frac{1}{\{1-u-\rho r-rz(1-\rho^2)+z^2(u+\rho r-\rho^2)\}^{\frac{1}{2}}} \quad .$$

Differentiating partially with respect to u, we get

$$\frac{\partial}{\partial u}\left[M(u-rT,T)\,\frac{\partial T}{\partial z}\right] \sim (1-z^2)^{3/2}\,(1-2rz+z^2)^{\frac{n}{2}-2}\left[\frac{\partial}{\partial u}\,(1+\rho^2-2\rho r-2u)^{-\left(\frac{n-4}{2}\right)}\varphi_1^{-\frac{1}{2}}\varphi_2^{-\frac{1}{2}}\right.$$

$$+\frac{\partial}{\partial u}\,\{1+\rho^2-\rho r-u-rz(1+\rho^2)+z^2(\rho r+u)\}^{-\frac{1}{2}}\,\varphi_3^{-\left(\frac{n-4}{2}\right)}\,\varphi_2^{-\frac{1}{2}}$$

$$\left.+\frac{\partial}{\partial u}\,\{1-u-\rho r-rz(1-\rho^2)+z^2(u+\rho r-\rho^2)\}^{-\frac{1}{2}}\,\varphi_3^{-\left(\frac{n-4}{2}\right)}\,\varphi_1^{-\frac{1}{2}}\right]$$

where

$$\varphi_1 = 1+\rho^2-\rho r-u-rz(1+\rho^2)+z^2(\rho r+u)\ ,$$

$$\varphi_2 = 1-u-\rho r-rz(1-\rho^2)+z^2(u+\rho r-\rho^2)\ ,$$

$$\varphi_3 = 1+\rho^2-2\rho r-2u\ .$$

Thus

$$\frac{\partial}{\partial u}\left[M(u-rT,T)\,\frac{\partial T}{\partial z}\right] \sim (1-z^2)^{3/2}\,(1-2rz+z^2)^{\frac{n}{2}-2}\left[-\left(\frac{n-4}{2}\right)\varphi_3^{-\left(\frac{n-2}{2}\right)}(-2)\,\varphi_1^{-\frac{1}{2}}\,\varphi_2^{-\frac{1}{2}}\right.$$

$$-\tfrac{1}{2}\,\varphi_1^{-3/2}\,(-1+z^2)\,\varphi_3^{-\left(\frac{n-4}{2}\right)}\,\varphi_2^{-\frac{1}{2}}$$

$$\left.-\tfrac{1}{2}\,\varphi_2^{-3/2}\,(-1+z^2)\,\varphi_3^{-\left(\frac{n-4}{2}\right)}\,\varphi_1^{-\frac{1}{2}}\right]$$

$$\sim (1-z^2)^{3/2}(1-2rz+z^2)^{\frac{n}{2}-2}\,\varphi_1^{-\frac{1}{2}}\,\varphi_2^{-\frac{1}{2}}\,\varphi_3^{-\left(\frac{n-2}{2}\right)}\left[n-4+\tfrac{1}{2}(1-z^2)\varphi_3(\varphi_1^{-1}+\varphi_2^{-1})\right.$$

Hence $\quad \dfrac{\partial}{\partial u}\left[M(u-rT,T)\,\dfrac{\partial T}{\partial z}\right]\Bigg|_{u=o}$

$$\sim \frac{n(1-z^2)^{3/2}\,(1-2rz+z^2)^{\frac{n}{2}-2}\,\{1+\rho^2-2\rho r\}^{-(\frac{n-2}{2})}}{\{1+\rho^2-\rho r-rz(1+\rho^2)+\rho rz^2\}^{+\frac{1}{2}}\{1-\rho r-rz(1-\rho^2)+(\rho r-\rho^2)z^2\}^{+\frac{1}{2}}}$$

$$\times\Bigg[\,1-\frac{4}{n}$$

$$+\frac{1}{2n}\,\frac{(1-z^2)(1+\rho^2-2\rho r)(2-2\rho r+\rho^2-2rz+\overline{2\rho r-\rho^2}z^2)}{\{1+\rho^2-\rho r-rz(1+\rho^2)+\rho rz^2\}\{1-\rho r-rz(1-\rho^2)+(\rho r-\rho^2)z^2\}}\Bigg]\,.$$

Thus by equation (I.2), we have

$$h(r)\sim\frac{n}{2\pi i(1+\rho^2-2\rho r)^{\frac{n-2}{2}}}\int\varphi(z)\,(1-2rz+z^2)^{\frac{n}{2}-2}\,dz \tag{3.1.20}$$

where

$$\varphi(z)=\frac{(1-z^2)^{3/2}}{\{1+\rho^2-\rho r-rz(1+\rho^2)+\rho rz^2\}^{\frac{1}{2}}\{1-\rho r-rz(1-\rho^2)+(\rho r-\rho^2)z^2\}^{\frac{1}{2}}} \tag{3.1.21}$$

$$\times\left[1+\frac{1}{n}\left\{\frac{(1-z^2)(1+\rho^2-2\rho r)(1-\rho r+\frac{\rho^2}{2}-rz+\overline{\rho r-\frac{\rho^2}{2}}z^2)}{\{1+\rho^2-\rho r-rz(1+\rho^2)+\rho rz^2\}\{1-\rho r-rz(1-\rho^2)+(\rho r-\rho^2)z^2\}}-4\right\}\right]$$

As shown in Chapter II, Section 2, the transformation

$$z + \frac{1}{z} = \frac{1 + \rho^2 + 2rT}{\rho + T}$$

maps the T-plane cut exterior to the interval $\left[\frac{-(1+\rho)^2}{2(1+r)} , \frac{(1-\rho)^2}{2(1-r)} \right]$ on the real axis onto the interior of the unit circle, $|z| = 1$, in the z-plane. Also the path of integration crosses the real axis at $z = \rho$ and terminates on the boundary of the unit circle at $e^{-i\theta}$ and $e^{i\theta}$, where $r = \cos\theta$.

The only possible singularities of the integrand of the integral of $h(r)$ occur if

$$\text{either} \quad \varphi_1(z) \equiv 1 + \rho^2 - \rho r - r(1+\rho^2)z + \rho r z^2 = 0 ,$$

$$\text{or} \quad \varphi_2(z) \equiv 1 - \rho r - r(1-\rho^2)z + \rho(r-\rho)z^2 = 0 .$$

<u>Discussion of $\varphi_1(z)$</u>

$$\begin{aligned} \text{Now} \quad \varphi_1(r) &= 1 + \rho^2 - \rho r - r^2(1+\rho^2) + \rho r^3 \\ &= (1+\rho^2)(1-r^2)-\rho r(1-r^2) \\ &= (1-r^2)(1+\rho^2-\rho r) \\ &> 0 \text{, since } 1 - r^2 > 0 \text{ and } 1 - \rho r > 0 \text{ implies} \\ &\qquad 1 + \rho^2 - \rho r > 0 \end{aligned}$$

$$\begin{aligned} \varphi_1(\rho) &= 1 + \rho^2 - \rho r - r(1+\rho^2)\rho + \rho r \rho^2 \\ &= 1 + \rho^2 - 2\rho r \\ &= (1-\rho r)^2 + \rho^2(1-r^2) \end{aligned}$$

> 0 since $(1-\rho r)^2 > 0$, $\rho^2 \geq 0$, $1 - r^2 > 0$ for $|r| < 1$. Now $\varphi_1'(z) = -r(1+\rho^2) + 2\rho r z = 0$ if $z = \frac{1+\rho^2}{2\rho}$.

But $|\frac{1+\rho^2}{2\rho}| = \frac{1+|\rho|^2}{2|\rho|} > 1$. Hence $\varphi_1'(z) = 0$ for $|z| > 1$.

Thus $\varphi_1'(z)$ does not vanish between ρ and r as $|\rho| < 1$, $|r| < 1$.

Since $\varphi_1(r) > 0$, $\varphi_1(\rho) > 0$ and $\varphi_1'(z) \neq 0$ for $|z| < 1$, we conclude that there is no real singularity between ρ and r.

Now if

$$\varphi_1(z) = \rho r z^2 - r(1+\rho^2)z + 1 + \rho^2 - \rho r = 0 \quad ,$$

$$z = \frac{r(1+\rho^2) \pm \sqrt{r^2(1+\rho^2)^2 - 4\rho r(1+\rho^2-\rho r)}}{2\rho r}$$

$$= \frac{1+\rho^2}{2\rho} \pm \frac{\sqrt{r^2(1+\rho^2)^2 - 4\rho r(1+\rho^2-\rho r)}}{2\rho r} \quad .$$

The roots will be imaginary if $r^2(1+\rho^2)^2 - 4\rho r(1+\rho^2-\rho r) < 0$. In this case $|\mathrm{Re}\ z| = |\frac{1+\rho^2}{2\rho}| = \frac{1+|\rho|^2}{2|\rho|} > 1$. Hence even if there is a complex root it lies outside the unit circle. We, therefore, conclude that $\varphi_1(z)$ does not vanish for any z in the closed interval between ρ and r.

<u>Discussion of $\varphi_2(z)$</u>

Now $$\begin{aligned}\varphi_2(r) &= 1 - \rho r - r(1-\rho^2)r + \rho(r-\rho)r^2 \\ &= 1 - \rho r - r^2 + r^3\rho \\ &= (1-\rho r)(1-r^2) \\ &> 0\ , \quad \text{since} \quad 1 - \rho r > 0 \quad \text{and} \quad 1 - r^2 > 0\ .\end{aligned}$$

$$\begin{aligned}\varphi_2(\rho) &= 1 - \rho r - r(1-\rho^2)\rho + \rho(r-\rho)\rho^2 \\ &= 1 - \rho r - \rho r + \rho^3 r + \rho^3 r - \rho^4 \\ &= 1 - \rho^4 - 2\rho r + 2\rho^3 r \\ &= (1-\rho^2)(1+\rho^2-2\rho r) \\ &> 0, \quad \text{since} \quad 1 - \rho^2 > 0 \quad \text{and} \quad 1 + \rho^2 - 2\rho r > 0 \\ &\qquad \text{for} \quad |r| < 1, \quad |\rho| < 1\ .\end{aligned}$$

Now $$\varphi_2'(z) = -r(1-\rho^2) + 2\rho(r-\rho)z = 0 \quad \text{if} \quad z = \frac{r(1-\rho^2)}{2\rho(r-\rho)}$$

and $$\varphi_2''(z) = 2\rho(r - \rho)\ .$$

If $\varphi_2'(z) = 0$ and $\varphi_2''(z) < 0$, then there is a maximum between ρ and r. Hence there would not be a singularity between ρ and r due to $\varphi_2(z)$ as $\varphi_2(r) > 0$ and $\varphi_2(\rho) > 0$.

There might however be a singularity if there were a minimum for z between ρ and r, which is so

$$\text{if} \qquad \varphi_2''(z) > 0$$

$$\text{or if} \qquad \rho(r-\rho) > 0$$

$$\text{or if either} \qquad \rho > 0, \quad r - \rho > 0 \qquad \text{or} \quad \rho < 0, \quad r - \rho < 0 \quad .$$

Hence there could be a minimum if $r > \rho > 0$ or $r < \rho < 0$.

Case I $r > \rho > 0$

$$\begin{aligned}
\varphi_2'(\rho) &= 2\rho(r-\rho)\rho - r(1-\rho^2) \\
&= 2\rho^2 r - 2\rho^3 - r + r\rho^2 \\
&= - r - 2\rho^3 + r\rho^2 + \rho.2\rho r \\
&< - r - 2\rho^3 + r\rho^2 + \rho(1+\rho^2) \qquad [\text{Because} \quad 1+\rho^2 > 2\rho > 2\rho r] \\
&= - r + r\rho^2 + \rho - \rho^3 \\
&= (\rho - r)(1 - \rho^2) \\
&< 0 \qquad \text{since} \quad \rho < r \quad \text{and} \quad 1 - \rho^2 > 0 \quad .
\end{aligned}$$

$$\begin{aligned}
\text{And} \qquad \varphi_2'(r) &= 2\rho(r - \rho)r - r(1 - \rho^2) \\
&= 2\rho r^2 - 2\rho^2 r - r + r\rho^2 \\
&= - r(1 + \rho^2 - 2\rho r) \\
&< 0 \qquad \text{since} \quad r > 0 \ , \quad 1 + \rho^2 - 2\rho r > 0 \quad .
\end{aligned}$$

Hence $\varphi_2'(z)$ is negative for $z = \rho$ and $z = r$.

<u>Case II $r < \rho < 0$</u>

Let $r' = -r$ and $\rho' = -\rho$.

Now
$$\begin{aligned}
\varphi_2'(\rho) &= -r + 3r\rho^2 - 2\rho^3 \\
&= r' - 3r'\rho'^2 + 2\rho'^3 \\
&= r' - r'\rho'^2 + \rho'(-2r'\rho') + 2\rho'^3 \\
&> r' - r'\rho'^2 + \rho'\{-(1+\rho'^2)\} + 2\rho'^3 \qquad \left[\text{since } 1+\rho'^2 > 2\rho'r', \; -(1+\rho'^2) < -2\rho'r'\right] \\
&= r'(1-\rho'^2) - \rho' - \rho'^3 + 2\rho'^3 \\
&= (r'-\rho')(1-\rho'^2) \\
&> 0 \qquad \text{since } 1-\rho'^2 > 0 \text{ and } r'-\rho' > 0 .
\end{aligned}$$

Again
$$\begin{aligned}
\varphi_2'(r) &= 2\rho r^2 - r - r\rho^2 \\
&= -2\rho'r'^2 + r' + r'\rho'^2 \\
&= r'(1+\rho'^2 - 2\rho'r') \\
&> 0 \qquad \text{since } r' > 0 ,\; 1+\rho'^2 - 2\rho'r' > 0 .
\end{aligned}$$

Hence $\varphi_2'(z) > 0$ for $z = \rho$ and $z = r$.

Cases I and II show that $\varphi_2'(z)$ does not vanish between $z = \rho$ and $z = r$. Hence there cannot be a minimum between $z = \rho$ and r. We, therefore, conclude that there is no real singularity in the closed interval between ρ and r due to $\varphi_2(z)$.

Now if

$$\varphi_2(z) = \rho(r-\rho)z^2 - r(1-\rho^2)z + 1 - \rho r = 0 \quad ,$$

$$z = \frac{+ r(1-\rho^2) \pm \sqrt{r^2(1-\rho^2)^2 - 4\rho(r-\rho)(1-\rho r)}}{2\rho(r - \rho)} \quad .$$

The roots will be imaginary if $r^2(1-\rho^2)^2 - 4\rho(r-\rho)(1-\rho r) < 0$ which is possible only if $\rho(r-\rho) > 0$.

If there are imaginary roots let them be $x \pm iy$. Obviously

$$x = \frac{r(1-\rho^2)}{2\rho(r-\rho)} \quad , \qquad y = \frac{\sqrt{4\rho(r-\rho)(1-\rho r)-r^2(1-\rho^2)^2}}{2\rho(r-\rho)} \quad .$$

Therefore $$|z|^2 = x^2 + y^2 = \frac{4\rho(r-\rho)(1-\rho r)}{4\rho^2(r-\rho)^2} = \frac{1 - \rho r}{\rho(r-\rho)} \quad .$$

Hence $$|z|^2 > 1 \quad \text{if} \quad \frac{1 - \rho r}{\rho(r-\rho)} > 1$$

$$\text{or if} \quad 1 - \rho r > \rho(r-\rho)$$

$$\text{or if} \quad 1 - \rho r - \rho r + \rho^2 > 0$$

$$\text{or if} \quad 1 - 2\rho r + \rho^2 > 0$$

which we know is true.

Hence the complex roots, if any, lie outside the unit circle. We therefore conclude that there is no singularity between ρ and r due to $\varphi_2(z)$.

Because there are no singularities in the closed interval joining ρ and r, we may deform the path of integration in the z-plane to be the straight line joining $e^{-i\theta}$ to $e^{i\theta}$ and crossing the real axis at $z = r$.

As in Chapter II, section 2, we may use the substitution (2.2.29) for z on the path of integration so that equation (3.1.20) becomes

$$h(r) \sim \frac{n(1-r^2)^{\frac{n}{2}-\frac{3}{2}}}{2\pi\,(1+\rho^2-2\rho r)^{\frac{n}{2}-1}} \int_{-1}^{1} \varphi(z)\,(1-\omega^2)^{\frac{n}{2}-2}\,d\omega \quad .$$

Following the method of Daniels [5] discussed in Chapter II, we expand $\varphi(z)$ as a power series in $(z-r)$ and integrate with respect to ω. By using equation (2.2.33), it is seen that

$$\varphi(z) = \varphi(r) + i\omega(1-r^2)^{\frac{1}{2}}\varphi'(r) + \ldots + \frac{i^k\omega^k(1-r^2)^{\frac{k}{2}}}{k!}\varphi^{(k)}(r) + \ldots$$

and hence,

$$(3.1.21)\quad h(r) \sim \frac{n(1-r^2)^{\frac{n-3}{2}}}{2\pi\,(1-2\rho r+\rho^2)^{\frac{n}{2}-1}} \sum_{k=o}^{\infty} \frac{i^k(1-r^2)^{\frac{k}{2}}}{k!}\varphi^k(r) \int_{-1}^{1} \omega^k(1-\omega^2)^{\frac{n}{2}-2}\,d\omega \,.$$

For the case that k is an odd integer, the integrand is an odd function, and the integral vanishes. Therefore, we may write equation (3.1.21) as

$$(3.1.22)\quad h(r) \sim \frac{n}{2\pi}\,\frac{(1-r^2)^{\frac{n-3}{2}}}{(1-2\rho r+\rho^2)^{\frac{n}{2}-1}} \sum_{k=o}^{\infty} \frac{(-1)^k(1-r^2)^k\varphi^{2k}(r)}{(2k)!} \int_{-1}^{1} \omega^{2k}(1-\omega^2)^{\frac{n}{2}-2}\,d\omega \,.$$

Since

$$I_k = \int_{-1}^{1} \omega^{2k} (1-\omega^2)^{\frac{n}{2}-2} \, d\omega$$

is the same as equation (2.2.36), the value of I_k is given by (2.2.37). Substituting this value in the expression for $h(r)$, we have

$$h(r) \sim \frac{n}{2\pi} \frac{(1-r^2)^{\frac{n-3}{2}}}{(1-2\rho r+\rho^2)^{\frac{n}{2}-1}} \left[I_o \varphi(r) + \sum_{k=1}^{\infty} \frac{(-1)^k (1-r^2)^k}{(2k)!} \varphi^{(2k)}(r) \frac{[1.3.5. \quad \ldots \quad (2k-1)]}{(n-1)(n+1)\ldots(n+2k-3)} I \right.$$

$$= \frac{n}{2\sqrt{\pi}} \frac{(1-r^2)^{\frac{n-3}{2}}}{(1-2\rho r+\rho^2)^{\frac{n}{2}-1}} \frac{\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{n-1}{2})} \varphi(r) \left\{ 1 + \sum_{k=1}^{\infty} \frac{(-1)^k (1-r^2)^k \varphi^{(2k)}(r)}{2^k k!(n-1)(n+1)\ldots(n+2k-3)} \right\} ,$$

where

$$(3.1.23) \quad \varphi(r) = \frac{(1-r^2)^{\frac{1}{2}}}{(1-\rho r)^{\frac{1}{2}}(1+\rho^2-\rho r)^{\frac{1}{2}}} \left\{ 1 + \frac{1}{n} \left[\frac{(1+\rho^2-2\rho r)(1-\rho r+\frac{\rho^2}{2})}{(1+\rho^2-\rho r)(1-\rho r)} - 4 \right] \right\} .$$

Hence

$$h(r) \sim \frac{n}{2\sqrt{\pi}} \frac{(1-r^2)^{\frac{n-3}{2}}}{(1+\rho^2-2\rho r)^{\frac{n-2}{2}}} \frac{\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{n-1}{2})} \varphi(r) \left[1 - \frac{(1-r^2)}{2(n-1)} \frac{\varphi''(r)}{\varphi(r)} + O(n^{-2}) \right] .$$

Let

$$(3.1.24) \qquad \psi(r) = \frac{(1-r^2)^{\frac{1}{2}}}{(1+\rho^2-\rho r)^{\frac{1}{2}}(1-\rho r)^{\frac{1}{2}}} ,$$

(3.1.25) $$g(r) = \frac{(1+\rho^2-2\rho r)(1-\rho r + \frac{\rho^2}{2})}{(1+\rho^2-\rho r)(1-\rho r)} - 4$$

and

(3.1.26) $$z(r) = \varphi(r)\left[1 - \frac{(1-r^2)}{2(n-1)}\,\frac{\varphi''(r)}{\varphi(r)} + O(n^{-2})\right] .$$

Then

$$z(r) = \psi(r)\left[1 + \frac{g(r)}{n}\right] - \frac{(1-r^2)}{2n}\,\varphi''(r) + O(n^{-2})$$

since $$\frac{1}{n-1} = \frac{1}{n(1-\frac{1}{n})} = \frac{1}{n}(1-\frac{1}{n})^{-1} = \frac{1}{n}(1 + \frac{1}{n} + \frac{1}{n^2} \ldots)$$

$$= \frac{1}{n} + O(n^{-2}) .$$

Or

(3.1.27) $$z(r) = \psi(r)\left[1 + \frac{g(r) - \{(1-r^2)\,\varphi''(r)\}/2\psi(r)}{n} + O(n^{-2})\right] .$$

Let

(3.1.28) $$t(r) = g(r) - \frac{(1-r^2)\,\varphi''(r)}{2\psi(r)} .$$

Now

$$t(r) = t(\rho) + t'(\rho)\,(r-\rho) + \frac{1}{2!}\,t''(\rho)(r-\rho)^2 + \ldots$$

$$= t(\rho) + t'(\rho)\,O(n^{-\frac{1}{2}}) .$$ [Using (III.7)] .

Hence

$$z(r) = \psi(r)\left[1 + \frac{t(\rho) + t'(\rho)\,O(n^{-\frac{1}{2}})}{n} + O(n^{-2})\right]$$

$$= \psi(r)\left[1 + \frac{t(\rho)}{n} + O(n^{-3/2})\right]$$

$$= c\psi(r)\left[1 + O(n^{-3/2})\right] , \quad \text{where} \quad c = 1 + \frac{t(\rho)}{n} .$$

Hence

$$h(r) \sim K \frac{(1-r^2)^{\frac{n}{2}-1}}{(1+\rho^2-2\rho r)^{\frac{n}{2}-1}(1+\rho^2-\rho r)^{\frac{1}{2}}(1-\rho r)^{\frac{1}{2}}} \left[1 + O(n^{-3/2})\right] , \tag{3.1.29}$$

where K is an adjusted normalizing constant.

Let

$$f(r) = [(1+\rho^2-\rho r)(1-\rho r)]^{-\frac{1}{2}} . \tag{3.1.30}$$

Then

$$\ell n\ f(r) = -\tfrac{1}{2}\,[\ell n\ (1+\rho^2-\rho r) + \ell n(1-\rho r)] ,$$

and

$$\frac{f'(r)}{f(r)} = -\frac{1}{2}\left[\frac{-\rho}{1+\rho^2-\rho r} + \frac{-\rho}{1-\rho r}\right]$$

$$= \frac{\rho}{2}\left[\frac{1-\rho r+1+\rho^2-\rho r}{(1+\rho^2-\rho r)(1-\rho r)}\right]$$

$$= \frac{1}{2}\ \frac{\rho(2-2\rho r+\rho^2)}{(1-\rho r+\rho^2)(1-\rho r)} .$$

$$\therefore\quad f'(r) = \frac{1}{2}\,\frac{\rho(2-2\rho r+\rho^2)}{\{(1+\rho^2-\rho r)(1-\rho r)\}^{3/2}} . \tag{3.1.31}$$

Taking logarithms, we get

$$\ell n\ f'(r) = \ell n\ \frac{\rho}{2} + \ell n(2-2\rho r+\rho^2) - \frac{3}{2}\ \ell n(1+\rho^2-\rho r) - \frac{3}{2}\ \ell n(1-\rho r) .$$

Differentiating with respect to r, we get

$$\frac{f''(r)}{f'(r)} = \frac{-2\rho}{2-2\rho r+\rho^2} + \frac{3}{2}\frac{\rho}{1+\rho^2-\rho r} + \frac{3}{2}\frac{\rho}{1-\rho r} \quad . \tag{3.1.32}$$

From (3.1.30), (3.1.31) and (3.1.32), we get

$$f(\rho) = (1-\rho^2)^{-\frac{1}{2}} \quad , \quad f'(\rho) = \frac{\rho(2-\rho^2)}{2(1-\rho^2)^{3/2}}$$

and

$$f''(\rho) = \frac{\rho(2-\rho^2)}{2(1-\rho^2)^{3/2}} \left[\frac{-2\rho}{2-\rho^2} + \frac{3}{2}\rho + \frac{3}{2}\frac{\rho}{1-\rho^2} \right]$$

$$= \frac{\rho^2(2-\rho^2)}{4(1-\rho^2)^{3/2}} \left[\frac{-4+4\rho^2+6-9\rho^2+3\rho^4+6-3\rho^2}{(2-\rho^2)(1-\rho^2)} \right]$$

$$= \frac{\rho^2(8-8\rho^2+3\rho^4)}{4(1-\rho^2)^{5/2}} \quad .$$

Hence expanding about ρ we get

$$f(r) = \frac{1}{(1-\rho^2)^{\frac{1}{2}}} \left[1 + \frac{\rho(2-\rho^2)}{2(1-\rho^2)}(r-\rho) + \frac{\rho^2(8-8\rho^2+3\rho^4)}{8(1-\rho^2)^2}(r-\rho)^2 + O(n^{-3/2}) \right] \quad . \tag{3.1.33}$$

Consider

$$\varphi(r) = (1+\rho^2-2\rho r)^{-\alpha} \quad .$$

Differentiating with respect to r twice we get

$$\varphi'(r) = (-\alpha)(1+\rho^2-2\rho r)^{-\alpha-1}(-2\rho) = 2\rho\alpha(1+\rho^2-2\rho r)^{-\alpha-1} \quad ,$$

and

$$\varphi''(r) = 4\rho^2\alpha(\alpha+1)\ (1+\rho^2-2\rho r)^{-\alpha-2} \ ,$$

giving

$$\varphi(\rho) = (1-\rho^2)^{-\alpha} \quad , \quad \varphi'(\rho) = 2\rho\alpha(1-\rho^2)^{-\alpha-1}$$

and

$$\varphi''(\rho) = 4\rho^2\alpha(\alpha+1)\ (1-\rho^2)^{-\alpha-2} \quad .$$

Thus

$$(3.1.34)\quad \varphi(r) = \frac{1}{(1-\rho^2)^{\alpha}} \left[\ 1 + \frac{2\rho\alpha}{1-\rho^2}\,(r-\rho) + \frac{2\rho^2\alpha(\alpha+1)}{(1-\rho^2)^2}\,(r-\rho)^2 + O(n^{-3/2})\ \right] .$$

From (3.1.33) and (3.1.34) we get

$$(3.1.35)\quad f(r)\,\varphi(r) = \frac{1}{(1-\rho^2)^{\alpha+\frac{1}{2}}} \left[1 + \left\{ \frac{\rho(2-\rho^2)}{2(1-\rho^2)} + \frac{2\rho\alpha}{1-\rho^2} \right\} (r-\rho) \right.$$

$$+ \left\{ \frac{\rho^2(8-8\rho^2+3\rho^4)}{8(1-\rho^2)^2} + \frac{\rho^2(2-\rho^2)\alpha}{(1-\rho^2)^2} + \frac{2\rho^2\alpha(\alpha+1)}{(1-\rho^2)^2} \right\}(r-\rho)$$

$$\left. + \ O(n^{-3/2})\ \right] .$$

The coefficient of $(r-\rho)$ will be zero if

$$\frac{\rho(2-\rho^2)}{2(1-\rho^2)} + \frac{2\rho\alpha}{1-\rho^2} = 0 \ ,$$

giving

$$\alpha = \frac{\rho^2 - 2}{4} \quad .$$

Substituting this value of α in (3.1.35), we get

$$f(r)\ \varphi(r) = \frac{1}{(1-\rho^2)^{\frac{\rho^2}{4}}} \left[1 + \frac{\rho^2(r-\rho)^2}{8(1-\rho^2)^2} \left\{8-8\rho^2+3\rho^4+8(2-\rho^2)\left(\frac{\rho^2-2}{4}\right)+16\left(\frac{\rho^2-2}{4}\right)\left(\frac{\rho^2+2}{4}\right)\right\} + 0(n^{-3/2})\right]$$

$$= \frac{1}{(1-\rho^2)^{\frac{\rho^2}{4}}} \left[1 + \frac{\rho^2(r-\rho)^2}{8(1-\rho^2)^2} \left\{8-8\rho^2+3\rho^4-8+8\rho^2-2\rho^4+\rho^4-4\right\} + 0(n^{-3/2})\right]$$

$$= \frac{1}{(1-\rho^2)^{\frac{\rho^2}{4}}} \left[1 + \frac{\rho^2(\rho^4-2)}{4(1-\rho^2)^2}(r-\rho)^2 + 0(n^{-3/2})\right] .$$

$$(3.1.36)\quad \therefore\ f(r)\ \varphi(r) = \frac{1}{(1-\rho^2)^{\frac{\rho^2}{4}}} \left[1 + \frac{\rho^2(\rho^4-2)}{4(1-\rho^2)} \frac{(r-\rho)^2}{1-\rho^2} + 0(n^{-3/2})\right] .$$

Now let

$$(3.1.37)\qquad \chi(r) = \frac{1-r^2}{1-2\rho r+\rho^2} .$$

$$\therefore\ \ell n\ \chi(r) = \ell n\ (1-r^2) - \ell n(1-2\rho r+\rho^2) .$$

Differentiating with respect to r, we get

$$(3.1.38)\qquad \frac{\chi'(r)}{\chi(r)} = \frac{-2r}{1-r^2} + \frac{2\rho}{1-2\rho r+\rho^2} .$$

Differentiating with respect to r once again, we get

$$(3.1.39) \qquad \frac{\chi(r)\,\chi''(r) - [\chi'(r)]^2}{[\chi(r)]^2} = \frac{(1-r^2)(-2)-(-2r)(-2r)}{(1-r^2)^2} + \frac{2\rho\;2\rho}{(1-2\rho r+\rho^2)^2} .$$

From (3.1.37), (3.1.38) and (3.1.39) we get

$$\chi(\rho) = 1, \quad \chi'(\rho) = 0 \quad \text{and} \quad \chi''(\rho) = -\frac{2}{1-\rho^2} .$$

Hence

$$\chi(r) = 1 - \frac{2}{1-\rho^2}\,\frac{(r-\rho)^2}{2!} + O(n^{-3/2}) .$$

$$\therefore \qquad \left\{\frac{1-r^2}{1-2\rho r+\rho^2}\right\}^{\frac{\rho^2(2-\rho^4)}{4(1-\rho^2)}} = 1 - \frac{\rho^2(2-\rho^4)}{4(1-\rho^2)}\,\frac{(r-\rho)^2}{(1-\rho^2)} + O(n^{-3/2}) .$$

Using (3.1.36) we get

$$f(r)\,\varphi(r) = \frac{1}{(1-\rho^2)^{\frac{\rho^2}{4}}} \left\{\frac{1-r^2}{1-2\rho r+\rho^2}\right\}^{\frac{\rho^2(2-\rho^4)}{4(1-\rho^2)}} \left\{1 + O(n^{-3/2})\right\} .$$

Hence (3.1.29) gives us

$$(3.1.40) \quad h(r) \sim K' \,\frac{(1-r^2)^{\frac{n}{2}-1}}{(1-2\rho r+\rho^2)^{\frac{n}{2}-1}}\,(1+\rho^2-2\rho r)^{\frac{\rho^2-2}{4}} \left\{\frac{1-r^2}{1-2\rho r+\rho^2}\right\}^{\frac{\rho^2(2-\rho^4)}{4(1-\rho^2)}}$$

$$\times \left\{1 + O(n^{-3/2})\right\}$$

$$= \frac{K'\,(1-r^2)^{\frac{n}{2}-1+\frac{\rho^2(2-\rho^4)}{4(1-\rho^2)}}}{(1-2\rho r+\rho^2)^{\frac{n}{2}-1-\left(\frac{\rho^2-2}{4}\right)+\frac{\rho^2(2-\rho^4)}{4(1-\rho^2)}}} \left\{1 + O(n^{-3/2})\right\} ,$$

where, again, K' is an adjusted normalizing constant.

Now let

$$N = n + \frac{\rho^2(2-\rho^4)}{2(1-\rho^2)} \quad . \tag{3.1.41}$$

Hence

$$h(r) \sim \frac{K' \, (1-r^2)^{\frac{N}{2}-1}}{(1-2\rho r+\rho^2)^{\frac{N}{2}-\frac{\rho^2}{4}-\frac{1}{2}}} \left\{ 1 + O(n^{-3/2}) \right\} \quad . \tag{3.1.42}$$

To renormalize $h(r)$, we consider

$$I = \int_{-1}^{1} \frac{(1-r^2)^{\frac{N}{2}-1}}{(1+\rho^2-2\rho r)^{\frac{N}{2}-\frac{1}{2}-\frac{\rho^2}{4}}} \, dr \quad .$$

Let $u = \dfrac{1+r}{2}$.

Then

$$\begin{aligned} 1 - r^2 &= 1 - (2u-1)^2 \\ &= 1 - (4u^2 - 4u + 1) \\ &= 4u(1-u) \quad , \end{aligned}$$

and

$$\begin{aligned} 1+\rho^2-2\rho r &= 1 + \rho^2 - 2\rho(2u-1) \\ &= (1+\rho)^2 - 4\rho u \\ &= (1+\rho)^2 \left\{ 1 - \frac{4\rho}{(1+\rho)^2} \, u \right\} \quad . \end{aligned}$$

Hence

$$I = \int_0^1 2^{N-1}\, u^{\frac{N}{2}-1}\,(1-u)^{\frac{N}{2}-1}\,(1+\rho)^{-N+1+\frac{\rho^2}{2}} \left\{1 - \frac{4\rho}{(1+\rho)^2}\, u\right\}^{-\frac{N}{2}+\frac{1}{2}+\frac{\rho^2}{4}} du$$

$$= 2^{N-1}\,(1+\rho)^{-N+1+\frac{\rho^2}{2}} \int_0^1 u^{\frac{N}{2}-1}\,(1-u)^{\frac{N}{2}-1} \left\{1 - \frac{4\rho}{(1+\rho)^2}\, u\right\}^{-\frac{N}{2}+\frac{1}{2}+\frac{\rho^2}{4}} du$$

Thus (using [41], page 293, example number 1), we get

$$I = 2^{N-1}(1+\rho)^{-(N-1-\frac{\rho^2}{2})}\, \frac{\Gamma(\frac{N}{2})\,\Gamma(\frac{N}{2})}{\Gamma(N)}\, F\left(\frac{N}{2} - \frac{1}{2} - \frac{\rho^2}{4}, \frac{N}{2} ; N ; \frac{4\rho}{(1+\rho)^2}\right),$$

where $F(a,b;c;z)$ is the usual hypergeometric function.

Using the duplication formula for the gamma function, ([41], page 240), we have

$$\pi^{\frac{1}{2}}\,\Gamma(N) = 2^{N-1}\,\Gamma(\tfrac{N}{2})\,\Gamma(\tfrac{N+1}{2}) \quad ,$$

giving

$$\frac{\Gamma(\frac{N}{2})\,\Gamma(\frac{N}{2})}{\Gamma(N)} = \frac{\sqrt{\pi}\;\Gamma(\frac{N}{2})}{2^{N-1}\,\Gamma(\frac{N+1}{2})} \quad .$$

Hence,

$$I = \sqrt{\pi}\,(1+\rho)^{-(N-1-\frac{\rho^2}{2})}\, \frac{\Gamma(\frac{N}{2})}{\Gamma(\frac{N+1}{2})}\, F\left(\frac{N}{2} - \frac{1}{2} - \frac{\rho^2}{4}, \frac{N}{2} ; N ; \frac{4\rho}{(1+\rho)^2}\right) .$$

Therefore the renormalized density function is given by

$$(3.1.43)\quad h(r) \sim K(N,\rho)\ \frac{(1-r^2)^{\frac{N}{2}-1}}{(1+\rho^2-2\rho r)^{\frac{N}{2}-\frac{1}{2}-\frac{\rho^2}{4}}}\ \left\{1 + O(n^{-3/2})\right\}$$

where

$$(3.1.44)\quad K(N,\rho) = \frac{(1+\rho)^{N-1-\frac{\rho^2}{2}}\ \Gamma(\frac{N+1}{2})}{\sqrt{\pi}\ \Gamma(\frac{N}{2})\ F\left(\frac{N}{2}-\frac{1}{2}-\frac{\rho^2}{4}, \frac{N}{2} ; N ; \frac{4\rho}{(1+\rho)^2}\right)}$$

and

$$N = n + \frac{\rho^2(2-\rho^4)}{2(1-\rho^2)} \quad .$$

When $\rho = 0$, we have

$$K(N,\rho)\Big|_{\rho=0} = K(n,0)$$

$$= \frac{1}{\sqrt{\pi}}\ \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})}\ \frac{1}{F(\frac{n-1}{2}, \frac{n}{2} ; n ; 0)}$$

$$= \frac{1}{\sqrt{\pi}}\ \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})}$$

and

$$(3.1.45)\quad h(r) \sim \frac{1}{\sqrt{\pi}}\ \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})}\ (1-r^2)^{\frac{n}{2}-1}\ \left\{1 + O(n^{-3/2})\right\} \quad .$$

For $\rho = 0$, (2.2.42) gives us

$$K(N,\rho)\Big|_{=0} = \frac{1}{\sqrt{\pi}} \frac{\Gamma(\frac{n-1}{2}+1)}{\Gamma(\frac{n-1+1}{2})}$$

$$= \frac{1}{\sqrt{\pi}} \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})}$$

and from (2.2.41) we get the renormalized density function of Daniels' as

$$h(r) \sim \frac{1}{\sqrt{\pi}} \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})} (1-r^2)^{\frac{n-2}{2}} \left\{1 + O(n^{-3/2})\right\}$$

so that

$$(3.1.46) \quad h(r) \sim \frac{1}{\sqrt{\pi}} \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})} (1-r^2)^{\frac{n}{2}-1} \left\{1 + O(n^{-3/2})\right\} .$$

The identity of (3.1.45) and (3.1.46) shows that our result is the same as that of Daniels' for $\rho = 0$ and thus provides a sort of check.

CHAPTER IV

DISTRIBUTION OF AN ESTIMATE OF THE SERIAL CORRELATION COEFFICIENT UNKNOWN MEAN

We consider the same Pre-stationary Linear Markov Process as in Chapter III and discuss the case of the unknown mean in the present chapter. We estimate the unknown mean $\bar{x}$ by

$$(4.1.1) \qquad \bar{x} = \frac{\frac{1}{2}x_1+x_2+x_3+\dots+x_{n-1}+\frac{1}{2}x_n}{n-1}$$

and ρ the serial correlation coefficient is estimated by

$$(4.1.2) \qquad r = \frac{C}{C_o} \quad ,$$

where

$$(4.1.3) \qquad C = (x_1-\bar{x})(x_2-\bar{x})+(x_2-\bar{x})(x_3-\bar{x})+\dots+(x_{n-1}-\bar{x})(x_n-\bar{x})$$

and

$$(4.1.4) \qquad C_o = \tfrac{1}{2}(x_1-\bar{x})^2+(x_2-\bar{x})^2+\dots+(x_{n-1}-\bar{x})^2+\tfrac{1}{2}(x_n-\bar{x})^2 \quad .$$

Now

$$\begin{aligned} C &= (x_1-\bar{x})(x_2-\bar{x})+(x_2-\bar{x})(x_3-\bar{x})+\dots+(x_{n-1}-\bar{x})(x_n-\bar{x}) \\ &= x_1x_2 - \bar{x}(x_1+x_2) + \bar{x}^2 \\ &\quad + x_2x_3 - \bar{x}(x_2+x_3) + \bar{x}^2 \\ &\quad + \dots\dots\dots\dots \\ &\quad + x_{n-1}x_n - \bar{x}(x_{n-1}+x_n) + \bar{x}^2 \end{aligned}$$

$$= c - 2\bar{x}(\tfrac{1}{2}x_1 + x_2 + \ldots + x_{n-1} + \tfrac{1}{2}x_n) + (n-1)\bar{x}^2$$

$$= c - 2\bar{x}(n-1)\bar{x} + (n-1)\bar{x}^2$$

or

$$C = c - (n-1)\bar{x}^2 \tag{4.1.5}$$

and

$$C_o = \tfrac{1}{2}(x_1-\bar{x})^2 + (x_2-\bar{x})^2 + \ldots + (x_{n-1}-\bar{x})^2 + \tfrac{1}{2}(x_n-\bar{x})^2$$

$$= \tfrac{1}{2}x_1^2 - x_1\bar{x} + \tfrac{1}{2}\bar{x}^2$$

$$+ x_2^2 - 2x_2\bar{x} + \bar{x}^2$$

$$+ x_3^2 - 2x_3\bar{x} + \bar{x}^2$$

$$+ \ldots\ldots\ldots$$

$$+ x_{n-1}^2 - 2x_{n-1}\bar{x} + \bar{x}^2$$

$$+ \tfrac{1}{2}x_n^2 - x_n\bar{x} + \tfrac{1}{2}\bar{x}^2$$

$$= c_o - 2\bar{x}(\tfrac{1}{2}x_1 + x_2 + x_3 + \ldots + x_{n-1} + \tfrac{1}{2}x_n) + (n-1)\bar{x}^2$$

$$= c_o - 2\bar{x}(n-1)\bar{x} + (n-1)\bar{x}^2$$

or

$$C_o = c_o - (n-1)\bar{x}^2 \quad . \tag{4.1.6}$$

The joint moment-generating function of C and C_o is

$$M(T_o,T) = E(T_oC_o+TC)$$

$$= (2\pi)^{-\frac{n}{2}} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{T_oC_o+TC-\frac{1}{2}\underline{x}'\underline{C}^{-1}\underline{x}}\, dx_1dx_2\ldots dx_n$$

where $\underline{C}^{-1} = \begin{bmatrix} 1+\rho^2 & -\rho & 0 & \cdots & 0 & 0 \\ -\rho & 1+\rho^2 & -\rho & \cdots & 0 & 0 \\ 0 & -\rho & 1+\rho^2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1+\rho^2 & -\rho \\ 0 & 0 & 0 & \cdots & -\rho & 1 \end{bmatrix}$.

Now

$$\begin{aligned} T_oC_o + TC - \tfrac{1}{2}\,\underline{x}'\,\underline{C}^{-1}\,\underline{x} &= T_o(c_o-(n-1)\bar{x}^2) + T(c-(n-1)\bar{x}^2) - \tfrac{1}{2}\,\underline{x}'\,\underline{C}^{-1}\,\underline{x} \\ &= T_oc_o + Tc - \tfrac{1}{2}\,\underline{x}'\,\underline{C}^{-1}\,\underline{x} - (n-1)\bar{x}^2\,(T_o+T) \\ &= -\tfrac{1}{2}\,\underline{x}'\,\underline{B}\,\underline{x} - \frac{(T_o+T)}{n-1}\,\underline{x}'\,\underline{m}\,\underline{m}'\,\underline{x} \end{aligned}$$

where

$$\underline{B} = \begin{bmatrix} 1+\rho^2-T_o & -(\rho+T) & 0 & \cdots & 0 & 0 \\ -(\rho+T) & 1+\rho^2-2T_o & -(\rho+T) & \cdots & 0 & 0 \\ 0 & -(\rho+T) & 1+\rho^2-2T_o & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1+\rho^2-2T_o & -(\rho+T) \\ 0 & 0 & 0 & & -(\rho+T) & 1-T_o \end{bmatrix},$$

$$\underline{x}' = (x_1, x_2, \ldots, x_n)\,,$$

and $\underline{m}' = (\tfrac{1}{2}, 1, 1, \ldots, 1, \tfrac{1}{2})$.

Or

$$T_oC_o + TC - \tfrac{1}{2}\,\underline{x}'\,\underline{C}^{-1}\,\underline{x} = -\tfrac{1}{2}\,\underline{x}'\left[\underline{B} + \frac{2(T_o+T)}{n-1}\,\underline{m}\,\underline{m}'\right]\underline{x}\ .$$

Hence

$$M(T_o,T) = (2\pi)^{-n/2}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-\frac{1}{2}\,\underline{x}'\left[\underline{B} + \frac{2(T_o+T)}{n-1}\,\underline{m}\,\underline{m}'\right]\underline{x}}\, dx_1dx_2\ldots dx_n\ .$$

Hence

$$(4.1.7) \qquad M(T_o,T) = \left|\underline{B} + \frac{2(T_o+T)}{n-1}\,\underline{m}\,\underline{m}'\right|^{-\frac{1}{2}} .$$

Now

$$\underline{m}\,\underline{m}' = \begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 1 \\ \frac{1}{2} \end{bmatrix} \quad (\tfrac{1}{2},1,1,\ldots,1,\tfrac{1}{2})$$

$$= \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & 1 & 1 & \cdots & 1 & 1 & \frac{1}{2} \\ \frac{1}{2} & 1 & 1 & 1 & \cdots & 1 & 1 & \frac{1}{2} \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ \frac{1}{2} & 1 & 1 & 1 & \cdots & 1 & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} \end{bmatrix} .$$

Hence

$$(4.1.8) \qquad \frac{2(T_o+T)}{n-1}\,\underline{m}\,\underline{m}' = \begin{bmatrix} \frac{s}{2} & s & s & s & \cdots & s & s & \frac{s}{2} \\ s & 2s & 2s & 2s & \cdots & 2s & 2s & s \\ s & 2s & 2s & 2s & \cdots & 2s & 2s & s \\ s & 2s & 2s & 2s & \cdots & 2s & 2s & s \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ s & 2s & 2s & 2s & \cdots & 2s & 2s & s \\ \frac{s}{2} & s & s & s & \cdots & s & s & \frac{s}{2} \end{bmatrix}$$

where $\quad s = \dfrac{T_o+T}{n-1}$.

From (3.1.3) ,

$$(4.1.9)\quad \underline{B} = \begin{bmatrix} p+T_o & q & 0 & 0 & \cdots & 0 & 0 & 0 \\ q & p & q & 0 & \cdots & 0 & 0 & 0 \\ 0 & q & p & q & \cdots & 0 & 0 & 0 \\ 0 & 0 & q & p & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & p & q & 0 \\ 0 & 0 & 0 & 0 & \cdots & q & p & q \\ 0 & 0 & 0 & 0 & \cdots & 0 & q & 1-T_o \end{bmatrix} ,$$

where $p = 1 + \rho^2 - 2T_o$

and

$q = -(\rho+T)$.

Hence

$$(4.1.10)\quad |\underline{B} + \frac{2(T_o+T)}{n-1}\underline{m}\,\underline{m}'| = \begin{vmatrix} p+T_o+\frac{s}{2} & q+s & s & \cdots & s & s & \frac{s}{2} \\ q+s & p+2s & q+2s & \cdots & 2s & 2s & s \\ s & q+2s & p+2s & \cdots & 2s & 2s & s \\ s & 2s & q+2s & \cdots & 2s & 2s & s \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ s & 2s & 2s & \cdots & q+2s & 2s & s \\ s & 2s & 2s & \cdots & p+2s & q+2s & s \\ s & 2s & 2s & \cdots & q+2s & p+2s & q+s \\ \frac{s}{2} & s & s & \cdots & s & q+s & 1-T_o+\frac{s}{2} \end{vmatrix} .$$

This determinant is of the same form as given in Appendix II if we take $e = p + T_o$ and $f = 1 - T_o$.

Hence from (II.14) we have,

$$(4.1.11)\quad \left|\underline{B} + \frac{2(T_o+T)}{n-1}\,\underline{m}\,\underline{m}'\right| = \left[1 - 2s\left\{\frac{p+1+2q-n(p+2q)}{(p+2q)^2}\right\}\right] Q_n$$

$$+ \frac{s(p+2T_o)^2}{2(p+2q)^2} X_{n-1} + \frac{s(p+2T_o-2)^2}{2(p+2q)^2} Y_{n-1}$$

$$+ (-1)^{n-1} \frac{s(p+2T_o)(2-p-2T_o)}{(p+2q)^2} q^{n-1} \quad .$$

We observe that:

1) Q_n of equation (II.6) is the same as $|\underline{B}|$.

2) X_{n-1} of equation (II.12) is equal to $(-q)^{n-1} A_{n-1}$ where A_{n-1} is defined as in equation (3.1.8).

3) Y_{n-1} of equation (II.13) is equal to $(-q)^{n-1} A'_{n-1}$ where A'_{n-1} is obtained from A_{n-1} by substituting $e(=p+T_o)$ for $f(=1-T_o)$.

4) $p = 1+\rho^2-2T_o$ gives us b (as defined in equation (3.1.5)) equal to $-\frac{p}{q}$.

5) $q = -(\rho+T)$ gives us a (as defined in equation (3.1.5)) equal to $-\frac{1}{q}$.

Thus $q = -\frac{1}{a}$ and $p = -qb = \frac{b}{a}$.

We now proceed to evaluate X_{n-1}, Y_{n-1} and various other expressions so as to be able to evaluate the determinant in equation (4.1.10).

Since

$$X_{n-1} = (-q)^{n-1} A_{n-1}$$

$$= \frac{1}{a^{n-1}} A_{n-1}$$

from equations (2.2.20) and (3.1.16) we get

$$X_{n-1} = \frac{1}{a^{n-1}} \left[\frac{(1-z^{2n-2})\{(1+z^2)(1-u-\rho r)-rz(1-\rho^2)\}}{1+\rho^2-2\rho r-2u} - z^2(1-z^{2n-4}) \right] \frac{1}{z^{n-1}(1-z^2)} ,$$

or

$$(4.1.12) \quad X_{n-1} = \frac{(1+\rho^2-2\rho r-2u)^{n-2}}{(1-z^2)(1-2rz+z^2)^{n-1}}$$

$$\times [(1-z^{2n-2})\{(1+z^2)(1-u-\rho r)-rz(1-\rho^2)\}-z^2(1-z^{2n-4})(1+\rho^2-2\rho r-2u)] ,$$

and using equation (3.1.13) we get from (II.13),

$$Y_{n-1} = \frac{1}{a^{n-1}} \left[\frac{1-(b+aT_o)z}{1-z^2} z^{n-1} + \frac{(b+aT_o-z)z}{1-z^2} z^{-n+1} \right] .$$

Now using equations (2.2.22) and (3.1.11) we get

$$b+aT_o = \frac{1+z^2}{z} + \frac{(1+z^2)(u+\rho r)-rz(1+\rho^2)}{z(1+\rho^2-2\rho r-2u)}$$

$$= \frac{(1+z^2)(1+\rho^2-2\rho r-2u+u+\rho r)-rz(1+\rho^2)}{z(1+\rho^2-2\rho r-2u)}$$

or

$$(4.1.13) \quad b+aT_o = \frac{(1+z^2)(1+\rho^2-\rho r-u)\ -rz(1+\rho^2)}{z(1+\rho^2-2\rho r-2u)} .$$

Therefore

$$1-(b+aT_o)z = 1 - \left\{ \frac{(1+z^2)(1+\rho^2-\rho r-u)-rz(1+\rho^2)}{1+\rho^2-2\rho r-2u} \right\}$$

$$= \frac{1+\rho^2-2\rho r-2u-(1+z^2)(1+\rho^2-\rho r-u)+rz(1+\rho^2)}{1+\rho^2-2\rho r-2u}$$

or

$$(4.1.14) \quad 1-(b+aT_o)z = \frac{-u-\rho r+rz(1+\rho^2)-(1+\rho^2-\rho r-u)z^2}{1+\rho^2-2\rho r-2u}$$

and

$$(4.1.15) \quad b+aT_o-z = \frac{(1+z^2)(1+\rho^2-\rho r-u)-rz(1+\rho^2)-z^2(1+\rho^2-2\rho r-2u)}{z(1+\rho^2-2\rho r-2u)}$$

which gives

$$(4.1.16) \quad (b+aT_o-z)z = \frac{1+\rho^2-\rho r-u-rz(1+\rho^2)+(u+\rho r)z^2}{1+\rho^2-2\rho r-2u} \quad .$$

Hence

$$Y_{n-1} = \frac{-[u+\rho r-rz(1+\rho^2)+(1+\rho^2-\rho r-u)z^2]z^{2n-2}+[1+\rho^2-\rho r-u-rz(1+\rho^2)+(u+\rho r)z^2]}{(1+\rho^2-2\rho r-2u)(1-z^2)z^{n-1}\ a^{n-1}}$$

$$= \left[\frac{-\{u+\rho r-rz(1+\rho^2)+(1+\rho^2-\rho r-u)z^2\}z^{2n-2}+\{1+\rho^2-\rho r-u-rz(1+\rho^2)+(u+\rho r)z^2\}}{(1+\rho^2-2\rho r-2u)(1-z^2)z^{n-1}}\right]$$

$$\times \frac{z^{n-1}\ (1+\rho^2-2\rho r-2u)^{n-1}}{(1-2rz+z^2)^{n-1}}$$

or

$$(4.1.17) \quad Y_{n-1} = [-\{u+\rho r-rz(1+\rho^2)+(1+\rho^2-\rho r-u)z^2\}z^{2n-2}$$

$$+ \{1+\rho^2-\rho r-u-rz(1+\rho^2)+(u+\rho r)z^2\}](1+\rho^2-2\rho r-2u)^{n-2}$$

$$\times \frac{1}{(1-2rz+z^2)^{n-1}\ (1-z^2)} \quad .$$

Now

$$p + 2T_o = 1 + \rho^2 - 2T_o + 2T_o \tag{4.1.18}$$

$$= 1 + \rho^2 ,$$

$$p + 2T_o - 2 = -1 + \rho^2 , \tag{4.1.19}$$

$$p + 2q = \frac{b}{a} - \frac{2}{a}$$

$$= \frac{b - 2}{a}$$

$$= \frac{z(1+\rho^2-2\rho r-2u)}{1-2rz+z^2}\left(z + \frac{1}{z} - 2\right)$$

or

$$p + 2q = \frac{(1-z)^2(1+\rho^2-2\rho r-2u)}{1-2rz+z^2} \tag{4.1.20}$$

and, from (2.2.21) and (2.2.22),

$$s = \frac{T_o+T}{n-1}$$

$$= \frac{1}{n-1}\left[\frac{(1+z^2)(u+\rho r)-rz(1+\rho^2)}{1-2rz+z^2} + \frac{z(1+\rho^2-2\rho r-2u)-\rho(1-2rz+z^2)}{1-2rz+z^2}\right]$$

giving

$$s = \frac{u-\rho+\rho r+(1-r+\rho^2-2u-\rho^2 r)z + (u+\rho r-\rho)z^2}{(n-1)(1-2rz+z^2)} \quad . \tag{4.1.21}$$

Hence

$$(4.1.22)\quad \frac{1-(n-1)(p+2q)}{(p+2q)^2} = \frac{1-(n-1)\left(\frac{1-2z+z^2}{1-2rz+z^2}\right)(1+\rho^2-2\rho r-2u)}{\left(\frac{1-2z+z^2}{1-2rz+z^2}\right)^2 (1+\rho^2-2\rho r-2u)^2}$$

$$= \frac{[1-2rz+z^2-(n-1)(1-z)^2(1+\rho^2-2\rho r-2u)][1-2rz+z^2]}{(1-z)^4\,(1+\rho^2-2\rho r-2u)^2},$$

$$(4.1.23)\quad \frac{(-1)^{n-1}(p+2T_o)(p+2T_o-2)q^{n-1}}{(p+2q)^2}$$

$$= \frac{(-1)^{n-1}(1+\rho^2)(-1+\rho^2)(-1)^{n-1}z^{n-1}(1+\rho^2-2\rho r-2u)^{n-1}}{(1-2rz+z^2)^{n-1}\,\frac{(1-2z+z^2)^2}{(1-2rz+z^2)^2}\,(1+\rho^2-2\rho r-2u)^2}$$

$$= -\frac{(1-\rho^4)z^{n-1}(1+\rho^2-2\rho r-2u)^{n-3}}{(1-z)^4\,(1-2rz+z^2)^{n-3}},$$

$$(4.1.24)\quad \left[\frac{p+2T_o}{2(p+2q)}\right]^2 = \left[\frac{(1+\rho^2)\,(1-2rz+z^2)}{2(1-z)^2(1+\rho^2-2\rho r-2u)}\right]^2$$

$$= \frac{(1+\rho^2)^2\,(1-2rz+z^2)^2}{4(1-z)^4(1+\rho^2-2\rho r-2u)^2},$$

and

$$(4.1.25)\quad \left[\frac{p+2T_o-2}{2(p+2q)}\right]^2 = \left[\frac{-(1-\rho^2)(1-2rz+z^2)}{2(1-z)^2(1+\rho^2-2\rho r-2u)}\right]^2$$

$$= \frac{(1-\rho^2)^2\,(1-2rz+z^2)^2}{4(1-z)^4(1+\rho^2-2\rho r-2u)^2}.$$

Substituting from equations (3.1.18), (4.1.12), (4.1.17), (4.1.18), (4.1.19), (4.1.20), (4.1.21), (4.1.22), (4.1.23), (4.1.24) and (4.1.25) in equation (4.1.11) we get after simplification,

$$|\underline{B} + \frac{2(T_o+T)}{n-1}\ \underline{m}\ \underline{m}'| \sim \frac{(1+\rho^2-2\rho r-2u)^{n-3}}{(1-z^2)(1-2rz+z^2)^n}$$

$$\times\{1+\rho^2-\rho r-u-r(1+\rho^2)z+(u+\rho r)z^2\}\{1-\rho r-u-r(1-\rho^2)z+(u+\rho r-\rho^2)z^2\}$$

$$\times\left[1+\rho^2-2\rho r-2u + 2\left\{\frac{u-\rho+\rho r+(1-r-2u+\rho^2-\rho^2 r)z+(u-\rho+\rho r)z^2}{(1-z)^2}\right\}\right]$$

where terms which are relatively $O(n^{-1})$ have been neglected.

Now

$$1+\rho^2-2\rho r-2u + 2\left\{\frac{u-\rho+\rho r+(1-r-2u+\rho^2-\rho^2 r)z+(u-\rho+\rho r)z^2}{(1-z)^2}\right\}$$

$$= \frac{(1-\rho)^2(1-2rz+z^2)}{(1-z)^2} .$$

Hence

$$(4.1.26)\quad |\underline{B} + \frac{2(T_o+T)}{n-1}\ \underline{m}\ \underline{m}'| \sim \frac{(1+\rho^2-2\rho r-2u)^{n-3}}{(1-z)^2(1-z^2)(1-2rz+z^2)^{n-1}}$$

$$\times\{(1+\rho^2-\rho r-u)-r(1+\rho^2)z+(u+\rho r)z^2\}\{(1-\rho r-u)-r(1-\rho^2)z+(u+\rho r-\rho^2)z^2\}(1-\rho)^2 .$$

Therefore

$$(4.1.27)\quad M(u-rT,T) = M(T_o,T) = \left|\underline{B} + \frac{2(T_o+T)}{n-1}\,\underline{m}\,\underline{m}'\right|^{-\frac{1}{2}}$$

$$\sim \frac{(1-z)(1-z^2)^{\frac{1}{2}}\,(1-2rz+z^2)^{\frac{n-1}{2}}}{(1+\rho^2-2\rho r-2u)^{\frac{n-3}{2}}}$$

$$\times \frac{1}{\{(1+\rho^2-\rho r-u)-r(1+\rho^2)z+(u+\rho r)z^2\}^{+\frac{1}{2}}\{(1-\rho r-u)-r(1-\rho^2)z+(u+\rho r-\rho^2)z^2\}^{+\frac{1}{2}}(1-\rho)}$$

Combining equations (2.2.25) and (4.1.27) we get

$$(4.1.28)\quad M(u-rT,T)\,\frac{\partial T}{\partial z} \sim \frac{(1-z)(1-z^2)^{\frac{3}{2}}(1-2rz+z^2)^{\frac{n-5}{2}}}{(1-\rho)(1+\rho^2-2\rho r-2u)^{\frac{n-5}{2}}}$$

$$\times \frac{1}{\{(1+\rho^2-\rho r-u)-r(1+\rho^2)z+(u+\rho r)z^2\}^{\frac{1}{2}}\{(1-\rho r-u)-r(1-\rho^2)z+(u+\rho r-\rho^2)z^2\}^{\frac{1}{2}}}\,.$$

Differentiating equation (4.1.28) partially with respect to u and setting $u = 0$ we get

$$\left[\frac{\partial}{\partial u}\left\{M(u-rT,T)\,\frac{\partial T}{\partial z}\right\}\right]_{u=0}$$

$$\sim \frac{(1-z)(1-z^2)^{\frac{3}{2}}\,(1-2rz+z^2)^{\frac{n-5}{2}}}{1-\rho}$$

$$\times\Big[(n-5)(1+\rho^2-2\rho r)^{\frac{-n+3}{2}}\{1+\rho^2-\rho r-r(1+\rho^2)z+\rho rz^2\}^{-\frac{1}{2}}\{1-\rho r-r(1-\rho^2)z+\rho(r-\rho)z^2\}^{-\frac{1}{2}}$$

$$+\tfrac{1}{2}(1-z^2)(1+\rho^2-2\rho r)^{\frac{-n+5}{2}}\{1+\rho^2-\rho r-r(1+\rho^2)z+\rho rz^2\}^{-\frac{3}{2}}\{1-\rho r-r(1-\rho^2)z+\rho(r-\rho)z^2\}^{-\frac{1}{2}}$$

$$+\tfrac{1}{2}(1-z^2)(1+\rho^2-2\rho r)^{\frac{-n+5}{2}}\{1+\rho^2-\rho r-r(1+\rho^2)z+\rho rz^2\}^{-\frac{1}{2}}\{1-\rho r-r(1-\rho^2)z+\rho(r-\rho)z^2\}^{-\frac{3}{2}}\Big]$$

(4.1.29)

$$\sim\frac{n(1-z)(1-z^2)^{\frac{3}{2}}(1-2rz+z^2)^{\frac{n-5}{2}}}{(1-\rho)\,(1+\rho^2-2\rho r)^{\frac{n-3}{2}}}\{1+\rho^2-\rho r-r(1+\rho^2)z+\rho rz^2\}^{-\frac{1}{2}}\{1-\rho r-r(1-\rho^2)z+\rho(r-\rho)z^2\}^{-\frac{1}{2}}$$

$$\times\left[1-\frac{5}{n}+\frac{(1-z^2)(1+\rho^2-2\rho r)}{2n}\left\{\frac{1}{1+\rho^2-\rho r-r(1+\rho^2)z+\rho rz^2}+\frac{1}{1-\rho r-r(1-\rho^2)z+\rho(r-\rho)z^2}\right\}\right]$$

Hence

$$(4.1.30)\quad \bar{h}(r)\sim\frac{1}{2\pi i}\,\frac{n}{(1-\rho)(1+\rho^2-2\rho r)^{\frac{n-3}{2}}}\int\bar{\Phi}(z)\,(1-2rz+z^2)^{\frac{n-5}{2}}\,dz$$

where

$$(4.1.31)\quad \bar{\Phi}(z)=\frac{(1-z)(1-z^2)^{\frac{3}{2}}}{\{1+\rho^2-\rho r-r(1+\rho^2)z+\rho rz^2\}^{\frac{1}{2}}\{1-\rho r-r(1-\rho^2)z+\rho(r-\rho)z^2\}^{\frac{1}{2}}}$$

$$\times\left[1-\frac{5}{n}+\frac{(1-z^2)(1+\rho^2-2\rho r)}{2n}\left\{\frac{1}{1+\rho^2-\rho r-r(1+\rho^2)z+\rho rz^2}+\frac{1}{1-\rho r-r(1-\rho^2)z+\rho(r-\rho)z^2}\right\}\right]$$

Therefore

$$(4.1.32)\quad \bar{\Phi}(r) = \frac{(1-r)(1-r^2)^{\frac{3}{2}}}{\{1+\rho^2-\rho r-r(1+\rho^2)r+\rho r.r^2\}^{\frac{1}{2}}\{1-\rho r-r(1-\rho^2)r+\rho(r-\rho)r^2\}^{\frac{1}{2}}}$$

$$\times\left[1-\frac{5}{n}+\frac{(1-r^2)(1+\rho^2-2\rho r)}{2n}\left\{\frac{1}{1+\rho^2-\rho r-r(1+\rho^2)r+\rho r.r^2}+\frac{1}{1-\rho r-r(1-\rho^2)r+\rho(r-\rho)r^2}\right\}\right.$$

$$= \frac{(1-r)(1-r^2)^{\frac{3}{2}}}{\{1-r^2+\rho^2(1-r^2)-\rho r(1-r^2)\}^{\frac{1}{2}}\{1-\rho r-r^2(1-\rho r)\}^{\frac{1}{2}}}\ \{1+O(n^{-1})\}$$

$$= \frac{(1-r)(1-r^2)^{\frac{3}{2}}}{(1-r^2)(1+\rho^2-\rho r)^{\frac{1}{2}}(1-\rho r)^{\frac{1}{2}}}\ \{1+O(n^{-1})\}$$

$$= \frac{(1-r)(1-r^2)^{\frac{1}{2}}}{(1+\rho^2-\rho r)^{\frac{1}{2}}(1-\rho r)^{\frac{1}{2}}}\ \{1+O(n^{-1})\}\quad .$$

As in Chapter II Section 2, we substitute in equation (2.2.35) for z on the path of integration and expand $\bar{\Phi}(z)$ as a power series in $z - r$ so that (4.1.30) becomes

$$(4.1.33)\quad \bar{h}(r) \sim \frac{n(1-r^2)^{\frac{n-4}{2}}}{2\pi(1-\rho)(1+\rho^2-2\rho r)^{\frac{n-3}{2}}}\sum_{k=o}^{\infty}\frac{\bar{\Phi}^{(2k)}(r)(-1)^k(1-r^2)^k}{(2k)!}\int_{-1}^{1}\omega^{2k}(1-\omega^2)^{\frac{n}{2}-\frac{5}{2}}\,d\omega$$

Now

$$I_k = \int_{-1}^{1} (\omega^2)^k \quad (1-\omega^2)^{\frac{n}{2}-\frac{5}{2}} \quad d\omega$$

$$= \int_{0}^{1} u^{k-\frac{1}{2}} \quad (1-u)^{\frac{n}{2}-\frac{5}{2}} \quad du$$

$$= \frac{\Gamma(\frac{2k+1}{2}) \quad \Gamma(\frac{n-3}{2})}{\Gamma(\frac{n+2k-2}{2})} \quad .$$

Hence

$$(4.1.34) \quad I_o = \frac{\Gamma(\frac{1}{2}) \quad \Gamma(\frac{n-3}{2})}{\Gamma(\frac{n-2}{2})} = \frac{\sqrt{\pi} \; \Gamma(\frac{n-3}{2})}{\Gamma(\frac{n-2}{2})} \quad .$$

Therefore

$$I_k = \frac{1.3.5. \quad \ldots \quad (2k-3)(2k-1)}{(n-2)(n)(n+2)\ldots(n+2k-4)} \quad I_o \quad .$$

Using an argument similar to the one used for obtaining (3.1.29), we get

$$(4.1.35) \quad \bar{h}(r) \sim \bar{K} \quad \frac{(1-r)(1-r^2)^{\frac{n-3}{2}}}{(1+\rho^2-2\rho r)^{\frac{n-3}{2}} \; (1+\rho^2-\rho r)^{\frac{1}{2}}(1-\rho r)^{\frac{1}{2}}} \quad \{1+O(n^{-\frac{3}{2}})\} \quad ,$$

where $\bar{K}$ is an adjusted normalizing constant.

Proceeding as in Chapter III for obtaining (3.1.40), we get

$$(4.1.36)\quad \bar{h}(r) \sim \bar{K}\,\frac{(1-r)(1-r^2)^{\frac{n-3}{2}}}{(1+\rho^2-2\rho r)^{\frac{n-3}{2}}}\;\frac{(1+\rho^2-2\rho r)^{\frac{\rho^2-2}{4}}}{(1-\rho^2)^{\frac{\rho^2}{4}}}\;\left\{\frac{1-r^2}{1+\rho^2-2\rho r}\right\}^{\frac{\rho^2(2-\rho^4)}{4(1-\rho^2)}}\{1+O(n^{-\frac{3}{2}})\}$$

$$= \frac{\bar{K}}{(1-\rho^2)^{\frac{\rho^2}{4}}}\quad\frac{(1-r)(1-r^2)^{\frac{n-3}{2}+\frac{\rho^2(2-\rho^4)}{4(1-\rho^2)}}}{(1+\rho^2-2\rho r)^{\frac{n-3}{2}-\frac{\rho^2-2}{4}+\frac{\rho^2(2-\rho^4)}{4(1-\rho^2)}}}\quad\{1+O(n^{-\frac{3}{2}})\}\;.$$

Hence

$$(4.1.37)\quad \bar{h}(r) \sim \bar{K}'\,\frac{(1-r)(1-r^2)^{\frac{N}{2}-\frac{3}{2}}}{(1+\rho^2-2\rho r)^{\frac{N}{2}-1-\frac{\rho^2}{4}}}\qquad\{1+O(n^{-\frac{3}{2}})\}$$

where

$$\bar{K}' = \frac{\bar{K}}{(1-\rho^2)^{\frac{\rho^2}{4}}}$$

and

$$N = n + \frac{\rho^2(2-\rho^4)}{2(1-\rho^2)}\qquad .$$

Consider

$$\bar{J}_n = \int_{-1}^{1}\frac{(1-r)(1-r^2)^{\frac{N-3}{2}}}{(1+\rho^2-2\rho r)^{\frac{N-2}{2}-\frac{\rho^2}{4}}}\,dr\qquad .$$

Substituting $u = \frac{1+r}{2}$ we get

$$\bar{J}_n = \int_0^1 \frac{2(1-u)\, 2^{\frac{N-3}{2}}\, u^{\frac{N-3}{2}}\, 2^{\frac{N-3}{2}}\, (1-u)^{\frac{N-3}{2}}\, 2du}{(1+\rho)^{N-2-\frac{\rho^2}{2}} \left[1 - \frac{4\rho}{(1+\rho)^2} u\right]^{\frac{N-2}{2} - \frac{\rho^2}{4}}}$$

$$= \frac{2^{N-1}}{(1+\rho)^{N-2-\frac{\rho^2}{2}}} \int_0^1 u^{\frac{N-3}{2}} (1-u)^{\frac{N-1}{2}} \left\{1 - \frac{4\rho}{(1+\rho)^2} u\right\}^{-(\frac{N-2}{2} - \frac{\rho^2}{4})} du$$

Using ([41], page 293, example number 1), we get

$$\bar{J}_n = \frac{2^{N-1}}{(1+\rho)^{N-2-\frac{\rho^2}{2}}} \frac{\Gamma(\frac{N-1}{2})\ \Gamma(\frac{N+1}{2})}{\Gamma(N)} F\left(\frac{N-2}{2} - \frac{\rho^2}{4}, \frac{N-1}{2}; N; \frac{4\rho}{(1+\rho)^2}\right) . \tag{4.1.38}$$

Using the duplication formula for the gamma function ([41], page 240) we have

$$\frac{\Gamma(\frac{N+1}{2})}{\Gamma(N)} = \frac{\sqrt{\pi}}{2^{N-1}\,\Gamma(\frac{N}{2})} .$$

Hence

$$\bar{J}_n = \frac{\sqrt{\pi}}{(1+\rho)^{N-2-\frac{\rho^2}{2}}} \frac{\Gamma(\frac{N-1}{2})}{\Gamma(\frac{N}{2})} F\left(\frac{N-2}{2} - \frac{\rho^2}{4}, \frac{N-1}{2}; N; \frac{4\rho}{(1+\rho)^2}\right) .$$

Therefore, the renormalized density function is

$$\bar{h}(r) \sim \bar{K}(N,\rho) \, \frac{(1-r)(1-r^2)^{\frac{N-3}{2}}}{(1+\rho^2-2\rho r)^{\frac{N-2}{2}-\frac{\rho^2}{4}}} \quad [1+O(n^{-\frac{3}{2}})] \tag{4.1.39}$$

where

$$\bar{K}(N,\rho) = \frac{(1+\rho)^{N-2-\frac{\rho^2}{2}} \quad \Gamma(\frac{N}{2})}{\sqrt{\pi}\ \Gamma(\frac{N-1}{2})\ F\left(\frac{N-2}{2}-\frac{\rho^2}{4}, \frac{N-1}{2} ; N ; \frac{4\rho}{(1+\rho)^2}\right)} \tag{4.1.40}$$

and $F(a,b;c;z)$ is the usual hypergeometric function.

When $\rho = 0$, we have

$$\bar{K}(N,\rho)\Big|_{\rho=o} = \bar{K}\ (n,o)$$

$$= \frac{1}{\sqrt{\pi}} \ \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})} \cdot \frac{1}{F\left(\frac{n-2}{2}, \frac{n-1}{2} ; n ; 0\right)}$$

$$= \frac{1}{\sqrt{\pi}} \ \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})}$$

and

$$\bar{h}(r) \sim \frac{1}{\sqrt{\pi}} \ \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})} \quad (1-r)(1-r^2)^{\frac{n-3}{2}} \quad [1+O(n^{-\frac{3}{2}})] \quad . \tag{4.1.41}$$

For $\rho = 0$, (2.2.43) gives us

$$\bar{h}(r) \sim \frac{2}{\sqrt{\pi}} \frac{\Gamma(\frac{n+2}{2})}{\Gamma(\frac{n-1}{2})\ (n-1+1)} (1-r)(1-r^2)^{\frac{n-1}{2}-1} [1+O(n^{-\frac{3}{2}})]$$

$$= \frac{1}{\sqrt{\pi}} \frac{2}{n} \frac{\frac{n}{2}\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})} (1-r)(1-r^2)^{\frac{n-3}{2}} [1+O(n^{-\frac{3}{2}})]$$

so that

$$\bar{h}(r) \sim \frac{1}{\sqrt{\pi}} \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})} (1-r)(1-r^2)^{\frac{n-3}{2}} [1+O(n^{-\frac{3}{2}})] \quad . \tag{4.1.42}$$

The identity of (4.1.41) and (4.1.42) shows that our result is the same as that of Daniels' for $\rho = o$ and thus provides a sort of check.

BIBLIOGRAPHY

[1] Anderson, R.L. "Distribution of the Serial Correlation Coefficient." *Ann. Math. Statistics* 13(1942), pp. 1-13.

[2] Anderson, R.L. and Anderson, T.W. "Distribution of the Circular Serial Correlation Coefficient for Residuals from a Fitted Fourier Series." *Ann. Math. Statistics* 21(1950), pp. 59-81.

[3] Chanda, K.C. "On Bounds of Serial Correlation." *Ann. Math. Statistics.* 33(1962), pp. 1457-1460.

[4] Daniels, H.E. "Saddle Point Approximations in Statistics." *Ann. Math. Statistics* 25(1954), pp. 631-650.

[5] Daniels, H.E. "The Approximate Distribution of Serial Correlation Coefficient." *Biometrika* 43(1956), pp. 169-185.

[6] Dixon, W.J. "Further Contributions to the Problem of Serial Correlation." *Ann. Math. Statistics* 15(1944), pp. 119-144.

[7] Durbin, J. "Testing for Serial Correlation in Systems of Simultaneous Regression Equations." *Biometrika* 44(1957), pp. 370-377.

[8] Durbin, J. and Watson, G.S. "Testing for Serial Correlation in Least Squares Regression I." *Biometrika* 37(1950), pp. 409-428.

[9] Durbin, J. and Watson, G.S. "Testing for Serial Correlation in Least Squares Regression II." *Biometrika* 38(1951), pp. 159-178.

[10] Erdelyi, Arthur; Magnus, Wilhelm; Oberhettinger, Fritz and Tricomi, Francesco G. Higher Transcendental Functions Volume I. Bateman Manuscript Project, McGraw Hill Book Company, New York (1953).

[11] Ghosh, M.N. "Asymptotic Distribution of Serial Statistics and Applications to Problems of Non-Parametric Tests of Hypothesis." Ann. Math. Statistics 25(1954), pp. 218-251.

[12] Gibson, G.D. "A Rapid Method for Ascertaining Serial Correlation." Biometrika 37(1950), pp. 288-307

[13] Griliches, Z. "A Note on Serial Correlation Bias in Estimates of Disturbed Lags." Econometrica 29(1961), pp. 65-73.

[14] Hannan, E.J. "Exact Tests for Serial Correlation." Biometrika 42(1955), pp. 133-142.

[15] Hannan, E.J. "Exact Tests for Serial Correlation in Vector Processes." Proc. Camb. Philos. Soc. 52(1956), pp. 482-487.

[16] Hannan, E.J. "Testing for Serial Correlation in Least Squares Regression." Biometrika 44(1957), pp. 57-66.

[17] Jenkins, G.M. "An Angular Transformation for the Serial Correlation Coefficient." Biometrika 41(1954), pp. 261-265.

[18] Kamat, A.R. and Sathe, Y.S. "Asymptotic Power of Certain Test Criteria (Based on First and Second Differences) for Serial Correlation Between Successive Observations." Ann. Math. Statistics 33(1962), pp. 187-201.

[19] Koopmans, Tjalling. "Serial Correlation and Quadratic Forms in Normal Variables." Ann. Math. Statistics 13(1942), pp. 14-33.

[20] Leipnik, R.B. "Distribution of the Serial Correlation Coefficient in a Circularly Correlated Universe." Ann. Math. Statistics 18(1947), pp. 80-87.

[21] Leipnik, R.B. "Moment Generating Functions of Quadratic Forms in Serially Correlated Normal Variables." Biometrika 45(1958), pp. 198-210.

[22] Madow, W.G. "Note on the Distribution of the Serial Correlation Coefficient." Ann. Math. Statistics 16(1945), pp. 308-310.

[23] Moran, P.A.P. "A Test for the Serial Independence of Residuals." Biometrika 37(1950), pp. 178-181.

[24] Ogawara, M. "A Note on the Test of Serial Correlation Coefficient." Ann. Math. Statistics 22(1951), pp. 115-118.

[25] Patton, A.C.A. "The Approximate Distribution of the Ratio of the Square Successive Difference to the Square Difference in Observations from a Stationary Normal Markov Process. M.Sc. Thesis, (Unpublished) University of Alberta, 1961.

[26] Quenouille, M. H. "The Joint Distribution of Serial Correlation Coefficients." Ann. Math. Statistics 20(1949), pp. 561-571.

[27] Rao, K. S. "Testing for Serial Correlation in a Stationary Multi-dimensional Discrete Stochastic Process." Bull. Inst. Internat. Statist. 24(1954), pp. 185-194.

[28] Rubin, H. "On the Distribution of the Serial Correlation Coefficient." Ann. Math. Statistics 16(1945), pp. 211-215.

[29] Sastry, A.S.R. "Bias in Estimation of Serial Correlation Coefficients." Sankhya 11(1951), pp. 281-296.

[30] Siddiqui, M. M. "Distribution of a Serial Correlation Coefficient Near the Ends of the Range." Ann. Math. Statistics. 29(1958), pp. 852-861.

[31] Walker, A. M. "The Asymptotic Distribution of Serial Correlation Coefficients for Autoregressive Process with Dependent Residuals." Proc. Camb. Philos. Soc. 50(1954) pp. 60-64.

[32] Watson, G. S. and Durbin, J. "Exact Tests of Serial Correlation Using Non-circular Statistics." Ann. Math. Statistics. 22(1951), pp. 446-451.

[33] Watson, G. S. "Serial Correlation in Regression Analysis I." Biometrika 42(1955), pp. 327-341.

[34] Watson, G. S. "On the Joint Distribution of the Circular Serial Correlation Coefficient." Biometrika 43(1956), pp. 161-168.

[35] Watson, G. S. and Hannan, E. J. "Serial Correlation in Regression Analysis II." Biometrika 43(1956), pp. 436-448.

[36] Weinstein, A. S. "Alternative Definitions of the Serial Correlation Coefficient in Short Autoregressive Sequences." J. Amer. Statist. Assoc. 53(1958), pp. 881-892.

[37] White, J. S. "Approximate Moments for the Serial Correlation Coefficient." Ann. Math. Statistics 28(1957), pp. 798-802.

[38] White, J. S. "The Limiting Distribution of the Serial Correlation Coefficient in the Explosive Case I." Ann. Math. Statistics 29(1958), pp. 1188-1197.

[39] White, J. S. "The Limiting Distribution of the Serial Correlation Coefficient in the Explosive Case II." Ann. Math. Statistics 30(1959), pp. 831-834.

[40] White, J. S. "Asymptotic Expansions for the Mean and Variance of the Serial Correlation Coefficient." Biometrika 48(1961), pp. 85-94.

[41] Whittaker, E.T. and Watson, G.N. A Course of Modern Analysis. Cambridge University Press, London, 4th Edition (1940).

APPENDIX I

SADDLE POINT APPROXIMATION

In the present appendix we discuss section 2 of Daniels' paper [5]. The distribution of statistics of the form

$$r = \frac{c}{c_o} \quad ,$$

where c_o is non-negative is to be considered. Let c_o, c have the joint probability density $f(c_o,c)$. We wish to find the distribution of r.

The Jacobian of the transformation

$$c = rc_o \;, \qquad c_o = c_o$$

is

$$\left| \frac{\partial(c_o,c)}{\partial(c_o,r)} \right| = c_o \quad .$$

Thus the joint probability density for c_o and r is

$$c_o \, f(c_o,rc_o) \quad .$$

The density for the distribution of r can be obtained as a marginal density by integrating out c_o to give

$$\text{(I.1)} \qquad h(r) = \int_0^\infty c_o \, f(c_o,rc_o) \, dc_o \quad .$$

Let $M(T_o,T)$ be the joint moment generating function of c_o and c. Then

$$M(T_o,T) = \mathcal{E}(e^{T_o c_o + Tc})$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{T_o c_o + Tc} f(c_o,c)\, dc_o\, dc \quad .$$

By the Fourier inversion formula we have

$$f(c_o,c) = \frac{1}{(2\pi i)^2} \int\int M(T_o,T)\, e^{-(T_o c_o + Tc)}\, dT_o\, dT \quad .$$

The paths of integration are along the imaginary axes in the T_o and T planes from $-i\infty$ to $+i\infty$ or along any allowable deformation of these paths. That is, along any path, $\alpha - i\infty$ to $\alpha + i\infty$, such that no singularities of $M(T_o,T)$ lie on the new path of integration or between it and the imaginary axis. Since $c = rc_o$, we have

$$f(c_o,rc_o) = \frac{1}{(2\pi i)^2} \int\int M(T_o,T)\, e^{-c_o(T_o + rT)}\, dT_o\, dT \quad .$$

Taking the linear transformation

$$u = T_o + rT \;, \quad T = T$$

with Jacobian

$$\left| \frac{\partial(T_o,T)}{\partial(u,T)} \right| = 1 \;,$$

we observe that this equation becomes

$$f(c_o, rc_o) = \frac{1}{(2\pi i)^2} \iint M(u-rT,T)\, e^{-c_o u}\, du\, dT \quad .$$

Then

$$\int_o^\infty f(c_o, rc_o) e^{c_o u}\, dc_o = \int_o^\infty \left[\frac{1}{(2\pi i)^2} \iint M(u-rT,T)\, e^{-c_o u}\, du\, dT \right] e^{c_o u}\, dc_o$$

$$= \frac{1}{2\pi i} \int \left\{ \int_o^\infty \left[\frac{1}{2\pi i} \int M(u-rT,T)\, e^{-c_o u}\, du \right] e^{c_o u}\, dc_o \right\} dT .$$

The Fourier-inversion formula gives

$$\int_o^\infty \left[\frac{1}{2\pi i} \int M(u-rT,T)\, e^{-c_o u}\, du \right] e^{c_o u}\, dc_o = M(u-rT,T)$$

so that

$$\int_o^\infty f(c_o, rc_o)\, e^{c_o u}\, dc_o = \frac{1}{2\pi i} \int M(u-rT,T)\, dT \quad .$$

Differentiating under the integral sign, where permissible, with respect to u, we obtain

$$\int_o^\infty f(c_o, rc_o) c_o e^{c_o u}\, dc_o = \frac{1}{2\pi i} \int \frac{\partial}{\partial u} [M(u-rT,T)]\, dT \quad .$$

Putting $u = o$ in this equation, we have, by equation (I.1)

$$h(r) = \frac{1}{2\pi i} \int \frac{\partial}{\partial u} [M(u-rT,T)] \bigg|_{u=o} dT$$

which is equation (2.5) of Daniels' paper [5].

If we wish to transform T to some other variable z by $T = T(z,u)$, then

$$M(u-rT,T)dT = M[u-rT(z,u),T(z,u)] \frac{\partial T(z,u)}{\partial z} dz .$$

For simplicity we write

$$M[u-rT(z,u),T(z,u)] \frac{\partial T(z,u)}{\partial z} dz = M(u-rT,T) \frac{\partial T}{\partial z} dz .$$

Then we have

$$\int_o^\infty f(c_o,rc_o)e^{c_o u} dc_o = \frac{1}{2\pi i} \int M(u-rT,T) \frac{\partial T}{\partial z} dz ,$$

where integration is along the transformed contour in the z-plane. By differentiating with respect to u under the sign of integration and setting $u = o$, we obtain

$$\text{(I.2)} \qquad h(r) = \frac{1}{2\pi i} \int \frac{\partial}{\partial u} [M(u-rT,T) \frac{\partial T}{\partial z}]\Big|_{u=o} dz .$$

This equation known as Cramer-Geary inversion formula, corresponds to equation (2.7) of Daniels' paper [5] and is the one used in Chapters II, III and IV.

The main problem of this paper is to evaluate approximately integrals of the above type when the statistics c and c_o are calculated from moderately large samples. The cases considered are found to have integrands which can be written in the form

$$\varphi(z)\ [\psi(z)]^n$$

where n is the sample size.

In the applications considered, $\varphi(z)$ may be expanded about $z = \hat{z}$ where

$$\psi'(\hat{z}) = 0$$

and the resulting series integrated term by term. It is found that this leads to an asymptotic expansion in powers of (n^{-1}) the dominant term of which is used as the approximation.

APPENDIX II

EVALUATION OF A DETERMINANT

We are interested in finding the value of a determinant of the form

$$
\text{(II.1)}\quad \mathcal{L}_n = \left|\begin{matrix}
e+\frac{s}{2} & q+s & s & \cdots & s & s & \frac{s}{2} \\
q+s & p+2s & q+2s & \cdots & 2s & 2s & s \\
s & q+2s & p+2s & \cdots & 2s & 2s & s \\
s & 2s & q+2s & \cdots & 2s & 2s & s \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\
s & 2s & 2s & \cdots & q+2s & 2s & s \\
s & 2s & 2s & \cdots & p+2s & q+2s & s \\
s & 2s & 2s & \cdots & q+2s & p+2s & q+s \\
\frac{s}{2} & s & s & \cdots & s & q+s & f+\frac{s}{2}
\end{matrix}\right|_n .
$$

This is the same as the $(n+1)x(n+1)$ determinant

$$
\text{(II.2)}\quad \mathcal{L}_n = \left|\begin{matrix}
1 & \frac{1}{2} & 1 & 1 & \cdots & 1 & 1 & \frac{1}{2} \\
0 & e+\frac{s}{2} & q+s & s & \cdots & s & s & \frac{s}{2} \\
0 & q+s & p+2s & q+2s & \cdots & 2s & 2s & s \\
0 & s & q+2s & p+2s & \cdots & 2s & 2s & s \\
0 & s & 2s & q+2s & \cdots & 2s & 2s & s \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\
0 & s & 2s & 2s & \cdots & q+2s & 2s & s \\
0 & s & 2s & 2s & \cdots & p+2s & q+2s & s \\
0 & s & 2s & 2s & \cdots & q+2s & p+2s & q+s \\
0 & \frac{s}{2} & s & s & \cdots & s & q+s & f+\frac{2}{s}
\end{matrix}\right|_{n+1} .
$$

Let $R_{ij}(x)$ stand for the operation of writing the elements of the i^{th} row after adding x-times the corresponding elements of the j^{th} row. Then performing the operations

$$R_{21}(-s),\ R_{31}(-2s),\ R_{41}(-2s),\ \ldots,\ R_{(n-1)1}(-2s),\ R_{n1}(-2s)$$

and $\quad R_{(n+1)1}(-s)$

we get

$$\text{(II.3)}\qquad \mathcal{L}_n = \left\|\begin{bmatrix} 1 & \frac{1}{2} & 1 & 1 & \ldots & 1 & 1 & \frac{1}{2} \\ -s & e & q & 0 & \ldots & 0 & 0 & 0 \\ -2s & q & p & q & \ldots & 0 & 0 & 0 \\ -2s & 0 & q & p & \ldots & 0 & 0 & 0 \\ -2s & 0 & 0 & q & \ldots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -2s & 0 & 0 & 0 & \ldots & q & 0 & 0 \\ -2s & 0 & 0 & 0 & \ldots & p & q & 0 \\ -2s & 0 & 0 & 0 & \ldots & q & p & q \\ -s & 0 & 0 & 0 & \ldots & 0 & q & f \end{bmatrix}\right\|_{n+1}$$

$$= \left\|\begin{bmatrix} 0 & \frac{1}{2} & 1 & 1 & \ldots & 1 & 1 & \frac{1}{2} \\ -s & e & q & 0 & \ldots & 0 & 0 & 0 \\ -2s & q & p & q & \ldots & 0 & 0 & 0 \\ -2s & 0 & q & p & \ldots & 0 & 0 & 0 \\ -2s & 0 & 0 & q & \ldots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -2s & 0 & 0 & 0 & \ldots & q & 0 & 0 \\ -2s & 0 & 0 & 0 & \ldots & p & q & 0 \\ -2s & 0 & 0 & 0 & \ldots & q & p & q \\ -s & 0 & 0 & 0 & \ldots & 0 & q & f \end{bmatrix}\right\|_{n+1}$$

$$+ \left\| \begin{bmatrix} 1 & \frac{1}{2} & 1 & 1 & \cdots & 1 & 1 & \frac{1}{2} \\ 0 & e & q & 0 & \cdots & 0 & 0 & 0 \\ 0 & q & p & q & \cdots & 0 & 0 & 0 \\ 0 & 0 & q & p & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & q & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & q & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & p & q & 0 \\ 0 & 0 & 0 & 0 & \cdots & q & p & q \\ 0 & 0 & 0 & 0 & \cdots & 0 & q & f \end{bmatrix} \right\|_{n+1}$$

or

$$\mathscr{L}_n = -2s\, P_{n+1} + Q_n \tag{II.4}$$

where

$$P_{n+1} = \left\| \begin{bmatrix} 0 & \frac{1}{2} & 1 & 1 & \cdots & 1 & 1 & \frac{1}{2} \\ \frac{1}{2} & e & q & 0 & \cdots & 0 & 0 & 0 \\ 1 & q & p & q & \cdots & 0 & 0 & 0 \\ 1 & 0 & q & p & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & q & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 1 & 0 & 0 & 0 & \cdots & q & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & p & q & 0 \\ 1 & 0 & 0 & 0 & \cdots & q & p & q \\ \frac{1}{2} & 0 & 0 & 0 & \cdots & 0 & q & f \end{bmatrix} \right\|_{n+1} \tag{II.5}$$

and

$$Q_n = \left\| \begin{bmatrix} e & q & 0 & \cdots & 0 & 0 & 0 \\ q & p & q & \cdots & 0 & 0 & 0 \\ 0 & q & p & \cdots & 0 & 0 & 0 \\ 0 & 0 & q & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & q & 0 & 0 \\ 0 & 0 & 0 & \cdots & p & q & 0 \\ 0 & 0 & 0 & \cdots & q & p & q \\ 0 & 0 & 0 & \cdots & 0 & q & f \end{bmatrix} \right\|_n . \tag{II.6}$$

Multiplying the sum of rows 2 to $(n+1)$ of P_{n+1} by $(-1)/(p+2q)$ and adding to the first row, we obtain

$$P_{n+1} = \left\| \begin{bmatrix} \frac{-n+1}{p+2q} & \frac{1}{2}-\frac{(e+q)}{p+2q} & 0 & 0 & \cdots & 0 & 0 & \frac{1}{2}-\frac{(q+f)}{p+2q} \\ \frac{1}{2} & e & q & 0 & \cdots & 0 & 0 & 0 \\ 1 & q & p & q & \cdots & 0 & 0 & 0 \\ 1 & 0 & q & p & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & q & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 & 0 & \cdots & q & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & p & q & 0 \\ 1 & 0 & 0 & 0 & \cdots & q & p & q \\ \frac{1}{2} & 0 & 0 & 0 & \cdots & 0 & q & f \end{bmatrix} \right\|_{n+1} .$$

Multiplying the sum of columns 2 to $n+1$ of P_{n+1} by $(-1)/(p+2q)$ and adding to the first column, we obtain

$$P_{n+1} = \left\| \begin{bmatrix} \frac{e+f+2q-n(p+2q)}{(p+2q)^2} & \frac{1}{2}-\frac{(e+q)}{p+2q} & 0 & 0 & \cdots & 0 & 0 & \frac{1}{2}-\frac{(q+f)}{p+2q} \\ \frac{1}{2}-\frac{(e+q)}{p+2q} & e & q & 0 & \cdots & 0 & 0 & 0 \\ 0 & q & p & q & \cdots & 0 & 0 & 0 \\ 0 & 0 & q & p & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & q & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & q & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & p & q & 0 \\ 0 & 0 & 0 & 0 & \cdots & q & p & q \\ \frac{1}{2}-\frac{(q+f)}{p+2q} & 0 & 0 & 0 & \cdots & 0 & q & f \end{bmatrix} \right\|_{n+1} ,$$

or

$$\text{(II.7)} \qquad P_{n+1} = \frac{e+f+2q-n(p+2q)}{(p+2q)^2} Q_n + \frac{2e-p}{2(p+2q)} M_n + (-1)^{n-1} \frac{2f-p}{2(p+2q)} N_n$$

where

$$\text{(II.8)} \qquad M_n = \left\| \begin{matrix} \frac{p-2e}{2(p+2q)} & 0 & 0 & \cdots & 0 & 0 & \frac{p-2f}{2(p+2q)} \\ q & p & q & \cdots & 0 & 0 & 0 \\ 0 & q & p & \cdots & 0 & 0 & 0 \\ 0 & 0 & q & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & q & 0 & 0 \\ 0 & 0 & 0 & \cdots & p & q & 0 \\ 0 & 0 & 0 & \cdots & q & p & q \\ 0 & 0 & 0 & \cdots & 0 & q & f \end{matrix} \right\|_n ,$$

and

$$\text{(II.9)} \qquad N_n = \left\| \begin{matrix} \frac{p-2e}{2(p+2q)} & 0 & 0 & \cdots & 0 & 0 & \frac{p-2f}{2(p+2q)} \\ e & q & 0 & \cdots & 0 & 0 & 0 \\ q & p & q & \cdots & 0 & 0 & 0 \\ 0 & q & p & \cdots & 0 & 0 & 0 \\ 0 & 0 & q & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & q & 0 & 0 \\ 0 & 0 & 0 & \cdots & p & q & 0 \\ 0 & 0 & 0 & \cdots & q & p & q \end{matrix} \right\|_n .$$

Now expanding in terms of the elements of the first row we get

$$\text{(II.10)} \qquad M_n = \frac{p-2e}{2(p+2q)} X_{n-1} + (-1)^{n-1} \frac{p-2f}{2(p+2q)} q^{n-1}$$

and

$$\text{(II.11)} \qquad N_n = \frac{p-2e}{2(p+2q)} q^{n-1} + (-1)^{n-1} \frac{p-2f}{2(p+2q)} Y_{n-1}$$

where

$$\text{(II.12)} \qquad X_{n-1} = \left| \begin{matrix} p & q & 0 & \cdots & 0 & 0 & 0 \\ q & p & q & \cdots & 0 & 0 & 0 \\ 0 & q & p & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & p & q & 0 \\ 0 & 0 & 0 & \cdots & q & p & q \\ 0 & 0 & 0 & \cdots & 0 & q & f \end{matrix} \right|_{n-1}$$

and

$$\text{(II.13)} \qquad Y_{n-1} = \left| \begin{matrix} e & q & 0 & \cdots & 0 & 0 & 0 \\ q & p & q & \cdots & 0 & 0 & 0 \\ 0 & q & p & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & p & q & 0 \\ 0 & 0 & 0 & \cdots & q & p & q \\ 0 & 0 & 0 & \cdots & 0 & q & p \end{matrix} \right|_{n-1}$$

Hence

$$\text{(II.14)} \qquad \mathcal{L}_n = \left[1 - 2s \left\{ \frac{e+f+2q-n(p+2q)}{(p+2q)^2} \right\} \right] Q_n + \frac{s(2e-p)^2}{2(p+2q)^2} X_{n-1} + \frac{s(2f-p)^2}{2(p+2q)^2} Y_{n-1} + (-1)^{n-1} \frac{s(2e-p)(2f-p)}{(p+2q)^2} q^{n-1} .$$

Now

$$\text{(II.15)} \qquad X_1 = f \quad ,$$

$$\text{(II.16)} \qquad X_2 = pf - q^2$$

and in general

$$X_{n-1} = p\,X_{n-2} - q^2\,X_{n-3}$$

or, equivalently,

(II.17) $$X_{n+2} - p\,X_{n+1} + q^2\,X_n = 0 \quad .$$

Taking E to be the forward difference operator of the calculus of finite differences defined as

(II.18) $$E\,X_j = X_{j+1} \quad ,$$

we may write equation (II.17) as

$$(E^2 - pE + q^2)X_n = 0 \quad .$$

Solving this difference equation, we find that the roots of the auxiliary equation

$$z^2 - pz + q^2 = 0$$

are $$u = \frac{p + \sqrt{p^2-4q^2}}{2}$$

and $$v = \frac{p - \sqrt{p^2-4q^2}}{2} \quad .$$

We observe that $u + v = p$ and $uv = q^2$. Hence the general solution of equation (II.17) is

(II.19) $$X_n = k_1\,u^n + k_2\,v^n \quad .$$

Applying conditions (II.15) and (II.16) to equation(II.17) we have

$$X_1 = k_1u + k_2v = f$$

and $$X_2 = k_1u^2 + k_2v^2 = pf - q^2 \quad .$$

Solving these equations we get

$$k_1 = \frac{(p-v)f - q^2}{u(u-v)}$$

and

$$k_2 = \frac{(u-p)f + q^2}{v(u-v)} \quad .$$

Substituting these values in (II.19) we get

$$X_n = \left[\frac{\{(p-v)f-q^2\}}{(u-v)} u^{n-1} + \frac{\{(u-p)f+q^2\}}{(u-v)} v^{n-1} \right] \quad .$$

Hence

(II.20) $$X_{n-1} = \left[\frac{\{(p-v)f-q^2\}}{(u-v)} u^{n-2} + \frac{\{(u-p)f+q^2\}}{(u-v)} v^{n-2} \right] \quad .$$

We observe that by interchange of rows and columns we can make Y_{n-1} take the form of X_{n-1} where instead of f we will have e . Hence

(II.21) $$Y_{n-1} = \frac{\{(p-v)e-q^2\}u^{n-2}}{(u-v)} + \frac{\{(u-p)e+q^2\}v^{n-2}}{(u-v)} \quad .$$

Again

(II.22) $$Q_n = eX_{n-1} - q^2X_{n-2} \quad .$$

Using (II.14), (II.20), (II.21) and (II.22) we get

$$\mathcal{L}_n = \left[1 - 2s\left\{ \frac{e+f+2q-n(p+2q)}{(p+2q)^2} \right\} \right] (eX_{n-1} - q^2X_{n-2})$$

$$+ \frac{s(2e-p)^2}{2(p+2q)^2} X_{n-1} + \frac{s(2f-p)^2}{2(p+2q)^2} Y_{n-1}$$

$$+ (-1)^{n-1} \frac{s(2e-p)(2f-p)}{(p+2q)^2} q^{n-1}$$

$$= \frac{(-1)^{n-1} s(2e-p)(2f-p)}{(p+2q)^2} q^{n-1}$$

$$+ \left[e - \frac{s}{2(p+2q)^2} \{4e[e+f+2q-n(p+2q)] - (2e-p)^2\}\right] X_{n-1}$$

$$+ s \frac{(2f-p)^2}{2(p+2q)^2} Y_{n-1}$$

$$- q^2 \left[1 - \frac{2s}{(p+2q)^2} \{e+f+2q-n(p+2q)\}\right] X_{n-2}$$

or

$$\mathcal{L}_n = \frac{(-1)^{n-1}}{(p+2q)^2} s(2e-p)(2f-p)q^{n-1}$$

$$+ \frac{1}{(p+2q)^2} \left[e(p+2q)^2 - \frac{s}{2} \{4e[e+f+2q-n(p+2q)] - (2e-p)^2\}\right]$$

$$\times \left[\frac{\{(p-v)f-q^2\}u^{n-2} + \{(u-p)f+q^2\}v^{n-2}}{u - v} \right]$$

$$+ \frac{s}{2} \left(\frac{2f-p}{p+2q}\right)^2 \left[\frac{\{(p-v)e-q^2\}u^{n-2} + \{(u-p)e+q^2\}v^{n-2}}{u - v}\right]$$

$$- \frac{q^2}{(p+2q)^2} \left[(p+2q)^2-2s\{e+f+2q-n(p+2q)\}\right]\left[\frac{\{(p-v)f-q^2\}u^{n-3}+\{(u-p)f+q^2\}v^{n-3}}{u - v}\right]$$

Hence

$$(II.22)\quad \mathcal{L}_n = \frac{ns}{2(p+2q)}\left[\{(p-v)f-q^2\}\frac{u^{n-3}}{u-v}(u-4q^2)+\{(u-p)f+q^2\}\frac{v^{n-3}}{u-v}(v-4q^2)\right]$$

$$+ (-1)^{n-1}\ \frac{s(2e-p)(2f-p)}{(p+2q)^2}\ q^{n-1}$$

$$+ \frac{1}{(p+2q)^2}\left[e(p+2q)^2-2se(e+f+2q)+\frac{s}{2}(2e-p)^2\right]$$

$$\times \left[\frac{\{(p-v)f-q^2\}u^{n-2}+\{(u-p)f+q^2\}v^{n-2}}{u-v}\right]$$

$$+ \frac{s}{2}\left(\frac{2f-p}{p+2q}\right)^2\left[\frac{\{(p-v)e-q^2\}u^{n-2}+\{(u-p)e+q^2\}v^{n-2}}{u-v}\right]$$

$$- \frac{q^2}{(p+2q)^2}\left[(p+2q)^2-2s(e+f+2q)\right]\left[\frac{\{(p-v)f-q^2\}u^{n-3}+\{(u-p)f+q^2\}v^{n-3}}{u-v}\right]$$

where u and v are the roots of the equation

$$x^2 - px + q^2 = 0 .$$

APPENDIX III

ACCURACY OF APPROXIMATION OF h(r)

From (3.1.24) we know that

$$h(r) \sim \frac{n}{2\sqrt{\pi}} \; \frac{(1-r^2)^{\frac{n-3}{2}}}{(1+\rho^2-2\rho r)^{\frac{n-2}{2}}} \; \frac{\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{n-1}{2})} \; \varphi(r) \left[1 - \frac{(1-r^2)}{2(n-1)} \frac{\varphi''(r)}{\varphi(r)} + O(n^{-2})\right]$$

where

$$\varphi(r) = \frac{(1-r^2)^{\frac{1}{2}}}{(1+\rho^2-\rho r)^{\frac{1}{2}}(1-\rho r)^{\frac{1}{2}}} \quad \left[1 + \frac{1}{n}\left\{\frac{(1+\rho^2-2\rho r)(1-\rho r+\frac{\rho^2}{2})}{(1+\rho^2-\rho r)(1-\rho r)} - 4\right\}\right].$$

Hence

$$h(r) \sim \frac{n}{2\sqrt{\pi}} \; \frac{(1-r^2)^{\frac{n-2}{2}}}{(1+\rho^2-2\rho r)^{\frac{n-2}{2}} \; (1+\rho^2-\rho r)^{\frac{1}{2}}(1-\rho r)^{\frac{1}{2}}} \; \frac{\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{n-1}{2})}$$

$$\times\left[1 - \frac{1-r^2}{2(n-1)} \; \frac{\varphi''(r)}{\varphi(r)} + O(n^{-2})\right] \left[1 + \frac{1}{n}\left\{\frac{(1+\rho^2-2\rho r)(1-\rho r+\frac{\rho^2}{2})}{(1+\rho^2-\rho r)(1-\rho r)} - 4\right\} + O(n^{-2})\right]$$

which gives us

$$h(\rho) \sim \frac{n}{2\sqrt{\pi}} \quad \frac{1}{(1-\rho^2)^{\frac{1}{2}}} \; \frac{\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{n-1}{2})} \quad \left[1 + O(n^{-1})\right] \qquad .$$

Now by Stirling's Formula [41] we know

$$\Gamma(x) = e^{-x} \, x^{x-\frac{1}{2}} \, (2\pi)^{\frac{1}{2}} \quad \left\{1 + O(x^{-1})\right\} \quad .$$

Thus

$$\frac{\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{n-1}{2})} = \frac{e^{-\frac{n}{2}+1}\,(\frac{n}{2}-1)^{\frac{n}{2}-\frac{3}{2}}\,(2\pi)^{\frac{1}{2}}\,\{1+O[(\frac{n}{2}-1)^{-1}]\}}{e^{-\frac{n-1}{2}}\,(\frac{n-1}{2})^{\frac{n}{2}-1}\,(2\pi)^{\frac{1}{2}}\,\{1+O[(\frac{n-1}{2})^{-1}]\}}$$

$$= e^{\frac{1}{2}}\,(\frac{n-2}{2})^{\frac{n-3}{2}}\,(\frac{2}{n-1})^{\frac{n}{2}-1}\,\{1+O(n^{-1})\}$$

$$= e^{\frac{1}{2}}\,(\frac{n-2}{n-1})^{\frac{n}{2}-1}\,(\frac{n-2}{2})^{-\frac{1}{2}}\,\{1+O(n^{-1})\}$$

$$= e^{\frac{1}{2}}\,[1-\frac{1}{n-1}]^{\frac{n-1}{2}}\,[1-\frac{1}{n-1}]^{-\frac{1}{2}}\,[\frac{n-2}{2}]^{-\frac{1}{2}}\,[1+O(n^{-1})]$$

$$\sim e^{\frac{1}{2}}\,e^{-\frac{1}{2}}\,[1-\frac{1}{n-1}]^{-\frac{1}{2}}\,n^{-\frac{1}{2}}\,[\frac{1}{2}-\frac{1}{n}]^{-\frac{1}{2}}\,[1+O(n^{-1})]\ ,$$

or

(III.1) $$\frac{\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{n-1}{2})} \sim n^{-\frac{1}{2}} \quad .$$

Hence

(III.2) $$h(r) \text{ at } r=\rho \text{ is } O(n^{\frac{1}{2}}) \quad .$$

To establish a bound on the order of magnitude of $\mathcal{E}[(r-\rho)^2]$, consider

$$\frac{(1-r^2)^{\frac{n}{2}-1}}{(1+\rho^2-2\rho r)^{\frac{n}{2}-1}\;(1+\rho^2-\rho r)^{\frac{1}{2}}(1-\rho r)^{\frac{1}{2}}}$$

$$= \left\{\frac{1-r^2}{1+\rho^2-2\rho r}\right\}^{\frac{n}{2}-1} \frac{1}{(1+\rho^2-\rho r)^{\frac{1}{2}}(1-\rho r)^{\frac{1}{2}}}$$

$$= \frac{1}{(1+\rho^2-\rho r)^{\frac{1}{2}}(1-\rho r)^{\frac{1}{2}}} \left\{1 - \frac{(r-\rho)^2}{1+\rho^2-2r\rho}\right\}^{\frac{n}{2}-1}$$

$$\Big[\text{since} \quad 1 - r^2 = 1+\rho^2-2\rho r-\rho^2+2\rho r-r^2$$

$$= 1+\rho^2-2\rho r-(r-\rho)^2\Big].$$

(III.3)
$$\leq \frac{1}{(1-|\rho|)^{\frac{1}{2}}(1+\rho^2-|\rho|)^{\frac{1}{2}}} \left\{1 - \frac{(r-\rho)^2}{1+\rho^2-2\rho r}\right\}^{\frac{n}{2}-1}$$

because

$$\rho r \leq |\rho r| = |\rho|\ |r| \leq |\rho|$$

giving

$$1 - \rho r \geq 1 - |\rho|$$

and
$$1 + \rho^2 - \rho r \geq 1 + \rho^2 - |\rho|$$

whence

$$\frac{1}{1-\rho r} \leq \frac{1}{1-|\rho|}$$

and
$$\frac{1}{1+\rho^2-\rho r} \leq \frac{1}{1+\rho^2-|\rho|} \quad .$$

Let

$$g(r) = \frac{n}{2\sqrt{\pi}} \; \frac{(1-r^2)^{\frac{n}{2}-1}}{(1+\rho^2-2\rho r)^{\frac{n}{2}-1} \; (1+\rho^2-\rho r)^{\frac{1}{2}}(1-\rho r)^{\frac{1}{2}}} \quad \frac{\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{n-1}{2})} \quad .$$

Then using (III.1) we get

$$\text{(III.4)} \qquad g(r) \lesssim O(n^{\frac{1}{2}}) \; \frac{1}{(1-|\rho|)^{\frac{1}{2}}} \; \frac{1}{(1+\rho^2-|\rho|)^{\frac{1}{2}}} \quad \left\{1 - \left(\frac{r-\rho}{1+|\rho|}\right)^2\right\}^{\frac{n}{2}-1} \quad ,$$

since
$$(1+|\rho|)^2 = 1+\rho^2+2|\rho|$$
$$> 1+\rho^2-2\rho r$$

giving
$$\frac{(r-\rho)^2}{(1+|\rho|)^2} < \frac{(r-\rho)^2}{1+\rho^2-2\rho r}$$

so that
$$1 - \frac{(r-\rho)^2}{1+\rho^2-2\rho r} < 1 - \left(\frac{r-\rho}{1+|\rho|}\right)^2 \qquad .$$

We know that

$$\text{(III.5)} \qquad 0 < \mathcal{E}[(r-\rho)^2] = \int_{-1}^{1} (r-\rho)^2 \, h(r) \, dr$$

$$\sim \int_{-1}^{1} (r-\rho)^2 \, g(r) \, dr$$

$$\leq O(n^{\frac{1}{2}}) \int_{-1}^{1} (r-\rho)^2 \left\{1 - \left(\frac{r-\rho}{1+|\rho|}\right)^2\right\}^{\frac{n}{2}-1} dr \quad .$$

Now

$$\int_{-1}^{1} (r-\rho)^2 \left\{1 - \left(\frac{r-\rho}{1+|\rho|}\right)^2\right\} dr$$

$$= \int_{\frac{-(1+\rho)}{1+|\rho|}}^{\frac{1-\rho}{1+|\rho|}} (1+|\rho|)^2 u^2(1-u^2)^{\frac{n}{2}-1} (1+|\rho|) \, du$$

$$= (1+|\rho|)^3 \int_{\frac{-(1+\rho)}{1+|\rho|}}^{\frac{1-\rho}{1+|\rho|}} u^2(1-u^2)^{\frac{n}{2}-1} du$$

$$< (1+|\rho|)^3 \int_{-1}^{1} u^2(1-u^2)^{\frac{n}{2}-1} du$$

$$= (1+|\rho|)^3 \frac{\Gamma(\frac{3}{2})\,\Gamma(\frac{n}{2})}{\Gamma(\frac{n+3}{2})}$$

$$\sim \frac{(\frac{n}{2})^{\frac{n}{2}-\frac{1}{2}} e^{-\frac{n}{2}} (2\pi)^{\frac{1}{2}}}{(\frac{n+3}{2})^{\frac{n+3}{2}-\frac{1}{2}} e^{-\frac{n+3}{2}} (2\pi)^{\frac{1}{2}}}$$

$$= \left(\frac{n}{n+3}\right)^{\frac{n}{2}} \left(\frac{n}{n+3}\right)^{-\frac{1}{2}} \left(\frac{2}{n+3}\right)^{\frac{3}{2}} e^{\frac{3}{2}}$$

$$= (1+\tfrac{3}{n})^{-\frac{n}{2}} (1+\tfrac{3}{n})^{+\frac{1}{2}} (\tfrac{n+3}{2})^{-\frac{3}{2}} \quad e^{\frac{3}{2}}$$

$$= O(n^{-\frac{3}{2}}) \quad .$$

This result combined with (III.5) suggests that the distribution of r is concentrated about $r = \rho$ within a range which is $O(n^{-\frac{1}{2}})$ on either side of ρ .

Therefore from (III.3)

$$\frac{(1-r^2)^{\frac{n}{2}-1}}{(1+\rho^2-2\rho r)^{\frac{n}{2}-1}(1+\rho^2-\rho r)^{\frac{1}{2}}(1-\rho r)^{\frac{1}{2}}} \leq \frac{1}{(1+\rho^2-|\rho|)^{\frac{1}{2}}(1-|\rho|)^{\frac{1}{2}}} \left[1 - \left(\frac{r-\rho}{1+|\rho|}\right)^2\right]^{\frac{n}{2}-1}$$

$$= \frac{1}{(1+\rho^2-|\rho|)^{\frac{1}{2}}(1-|\rho|)^{\frac{1}{2}}} \; e^{(\frac{n}{2}-1)\log\left\{1 - \left(\frac{r-\rho}{1+|\rho|}\right)^2\right\}}$$

$$= \frac{e^{(\frac{n}{2}-1)\left\{-\left(\frac{r-\rho}{1+|\rho|}\right)^2 - \frac{1}{2}\left(\frac{r-\rho}{1+|\rho|}\right)^4 - \cdots\right\}}}{(1+\rho^2-|\rho|)^{\frac{1}{2}}(1-|\rho|)^{\frac{1}{2}}}$$

the logarithmic expansion being valid because $|r - \rho| < 1 + |\rho|$ giving

$$\left(\frac{r - \rho}{1 + |\rho|}\right)^2 < 1 \quad .$$

Hence

$$\frac{(1-r^2)^{\frac{n}{2}-1}}{(1+\rho^2-2\rho r)^{\frac{n}{2}-1}\,(1+\rho^2-\rho r)^{\frac{1}{2}}(1-\rho r)^{\frac{1}{2}}} < \frac{e^{(\frac{n}{2}-1)\left\{-\left(\frac{r-\rho}{1+|\rho|}\right)^2\right\}}}{(1+\rho^2-|\rho|)^{\frac{1}{2}}(1-|\rho|)^{\frac{1}{2}}} .$$

Thus

$$\text{(III.6)} \qquad g(r) < O(n^{\frac{1}{2}})\; e^{-(\frac{n}{2}-1)\left(\frac{r-\rho}{1+|\rho|}\right)^2} .$$

Suppose $|r - \rho|$ is $O(n^{\frac{\alpha}{2}})$.

Then $\quad (\frac{n}{2} - 1)\left(\frac{r-\rho}{1+|\rho|}\right)^2$ is $O(n^{\alpha+1})$

and $\quad e^{-(\frac{n}{2} - 1)\left(\frac{r-\rho}{1+|\rho|}\right)^2}$

will tend to zero exponentially as $n \to \infty$ provided $\alpha > -1$.

Thus from (III.6) and (3.1.29) we see that $g(r)$ (and hence $h(r)$) is exponentially small for values of r outside some range, on either side of ρ, which is $O(n^{-\frac{1}{2}})$. We can conclude that over the effective range of r, $r - \rho$ may be considered to be of $O(n^{-\frac{1}{2}})$.